Intuitionistic Epistemic Logic in Coq

Final Bachelor Talk

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Context

- Motivation: Constructively analyze results about IEL (Artemov and Protopopescu, 2016)
- Epistemic logics try to model knowledge
- Modal operator K to model (propositional) knowledge (Hintikka)
- Here: Single agent perspective
- K K A the agent knows that the agent knows A
- Results interested in: soundness, completeness, decidability

IEL

How to give an account of knowledge faithful to BHK?

IEL

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- Intuitionistic knowledge is based on a verification (Artemov and Protopopescu, 2016; Williamson, 1982)
- $\mathsf{K}\,A$ is proven if one has conclusive verifiable evidence (certificate), which need not yield proof, that A is true.
- Examples for certificates:
  proofs, testimony of an authority, zero-knowledge proofs, .v files, classified sources
- Extends to empirical statements?

Accepting $A \supset \mathsf{K}\, A$

> *it expresses the trivial observation that, as soon as a proof of p is given, p becomes known.*
> *Martino and Usberti (1994)*

> *Suppose we are given a sentence [...] and a proof that it is true. Read the proof; thereby you come to know that the sentence is true. Reflecting on your recent learning you recognize that the sentence is now known by you; this shows that the truth is known.*
> *Bell and Hart (1979) p. 165*

- Not an omniscience claim!
- Its probably not that simple e.g. Williamson (1988) argues against this (proofs as types)

Rejecting $\mathsf{K}\,A \supset A$

- In classical logic expresses the facticity of knowledge
- Would need to have a uniform procedure transforming certificates into intuitionistic proofs.
- Can adopt different truth condition instead, e.g. $\mathsf{K}\,A \supset \neg\neg A$

	Classical	Intuitionistic
$A \supset \mathsf{K}\,(A)$	reject	endorse
$\mathsf{K}\,A \supset A$	endorse	reject

IEL: Formally

Formulas are generated by the following grammar:

$$A, B \ni \mathcal{F} := p_i \mid A \to B \mid A \wedge B \mid A \vee B \mid \mathsf{K}\, A \mid \bot \qquad (i \in \mathbb{N})$$

Definition (Axioms of IEL)

Axioms of IEL are the axioms of IPC and additionally

- $A \supset \mathsf{K}\, A$ (co-reflection)
- $\mathsf{K}\, A \supset \neg\neg A$ (intuitionistic reflection)
- $\mathsf{K}\,(A \supset B) \supset \mathsf{K}\, A \supset \mathsf{K}\, B$ (distribution)

K **and Coq**

K can be interpreted as propositional truncation.

- Sound embedding into Coq
- Perini Brogi (2021) suggests that IELs modality is weaker

Deduction system

Define natural deduction system $\vdash : \mathcal{L}(\mathcal{F}) \to \mathcal{F} \to \mathbb{P}$:

$$\frac{A \in \Gamma}{\Gamma \vdash A}\,\text{CTX} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,\text{II} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash A \to B}{\Gamma \vdash B}\,\text{IE}$$

$$\ldots$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \mathsf{K}\,A}\,\text{KR} \qquad \frac{\Gamma \vdash \mathsf{K}\,(A \to B)}{\Gamma \vdash \mathsf{K}\,A \to \mathsf{K}\,B}\,\text{KD} \qquad \frac{\Gamma \vdash \mathsf{K}\,A}{\Gamma \vdash \neg\neg A}\,\text{KT}$$

IEL := Logic of intuitionistic knowledge (with KT)

IEL^- := Logic of intuitionistic belief (without KT)

$$\mathcal{T} \vdash A :\Leftrightarrow \exists L.\, L \subseteq \mathcal{T} \wedge L \vdash A$$

Kripke Models for IEL, IEL$^-$

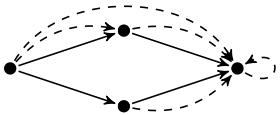

Figure: Model $\mathcal{M} = (\mathcal{W}, R, E, \mathcal{V})$

- Type of worlds $\mathcal{W}$
- Reachability relation $R : \mathcal{W} \to \mathcal{W} \to \mathbb{P}$
- Epistemic reachability relation $E : \mathcal{W} \to \mathcal{W} \to \mathbb{P}$
- Valuation: $\mathcal{V} : \mathcal{W} \to \mathbb{N} \to \mathbb{P}$

- $u \vDash \mathsf{K}\, A :\Leftrightarrow v \vDash A$ for all $v \in E(u)$
- $E \subseteq R$
- $R \circ E \subseteq E$ (shrink)
- IEL: $E(w) \neq \emptyset$

Results

Artemov and Protopopescu (2016)

- Soundness $\mathcal{T} \vdash A \to \mathcal{T} \Vdash A$
- strong completeness $\mathcal{T} \Vdash A \to \mathcal{T} \vdash A$ (classically)
- Completeness proof using canonical model construction with Lindenbaum Lemma

Our results

- Mechanization of results from paper
- Strong quasi-completeness: $\mathcal{T} \Vdash' A \to \neg\neg(\mathcal{T} \vdash A)$
- Completeness (using decidability): $\Gamma \Vdash' A \to \Gamma \vdash A$
- However soundness can only be proven using LEM.

Decidability

- Were not able to use e.g. finite model property.
- ND not well suited for proof search (no subformula property)
- Use sequent calculus (Krupski and Yatmanov, 2016) for proof search
- **2nd talk:** Use two different sequent calculi
  - one for cut-elimination (permutation)
  - one for decidability (membership)
- Obtain decider using a finite closure iteration (Dang, 2015; Menz, 2016; Smolka and Brown, 2012)

Cut-elimination proofs

- Idea: proof search in cut-free sequent calculus
- Usual cut-elimination proof (Troelstra and Schwichtenberg, 2000; Dragalin, 1987):
  - Introduce a depth-bounded system
  - Prove dp-weakening ($\Gamma \overset{h}{\Rightarrow} B \to A, \Gamma \overset{h}{\Rightarrow} B$)
  - Prove dp-inversion results
  - Prove dp-contraction ($A, A, \Gamma \overset{h}{\Rightarrow} B \to A, \Gamma \overset{h}{\Rightarrow} B$)
  - Prove cut using induction on pairs of numbers
- Dang (2015) and Smolka and Brown (2012)
  - No height-system, use a special sequent calculus
  - Prove weakening: $\Gamma \Rightarrow A \to \Gamma \subseteq \Omega \to \Omega \Rightarrow A$
  - Prove cut using 3 nested inductions

Two challenges:

- Can Dang and Smolka method be used for IEL?
- Do the results generalize to other modal logics?

Mixed-approach

We were able to prove cut using a mix of Dang & Smolka and Troelstra:

1. Use height-bounded variant of Dang-Smolka system for IEL
2. Prove dp-weakening: $\Gamma \overset{h}{\Rightarrow} A \to \Gamma \subseteq \Omega \to \Omega \overset{h}{\Rightarrow} A$
3. Prove inversion results
4. Prove cut using induction on pairs of natural numbers

Results:

- Much cleaner and less code (250 lines of code vs. 600 lines of code)
- Generalizes to classical modal logic K, using a sequent calculus by Hakli and Negri (2012).

Church-Fitch paradox (Fitch, 1963)

- The CF-paradox is an argument showing that from

$$A \supset \Diamond \mathsf{K}\, A \tag{WVER}$$

and

$$\exists A.\, A \wedge \neg \mathsf{K}\, A \tag{NOMN}$$

it is possible to derive

$$A \supset \mathsf{K}\, A$$

- Threat to verificationist theories of truth, since read classically this gives omniscience
- The Mystery of the Disappearing Diamond

Derivation of Church-Fitch (Brogaard and Salerno, 2019)

- Let A be the unknown truth. By WVER, $\Diamond \mathsf{K}\,(A \wedge \neg \mathsf{K}\, A)$.
- However $\neg \mathsf{K}\,(A \wedge \neg \mathsf{K}\, A)$ is a theorem (since knowledge is closed under conjunction).[1]
- Thus by necessitation, $\Box \neg \mathsf{K}\,(A \wedge \neg \mathsf{K}\, A)$ is a theorem.
- Using inter-definability of modal operators gives $\neg \Diamond \mathsf{K}\,(A \wedge \neg \mathsf{K}\, A)$
- Thus our assumption $\exists A.\, A \wedge \neg \mathsf{K}\, A$ is contradicted

[1] $\mathsf{K}\,(A \wedge B) \supset \mathsf{K}\, A \wedge \mathsf{K}\, B$

IEL response

- It is not important if derivation works in IEL (Artemov and Protopopescu, 2016)
- No paradox since IEL embraces the consequence
- Is an argument represented in logic vs. the derivation as an argument?
- In IEL K has a different reading - different knowledge
- Church-Fitch: K as collective knowledge - co-reflection?

Overview of contributions

- proof soundness and strong completeness for IEL using LEM
- constructive strong quasi-completeness and completeness but soundness under LEM; using modified semantics
- decidability + cut-elimination for K, IEL
- discussed relationship between IEL and two epistemic paradoxes (Fitch, 1963; Florio and Murzi, 2009)

Overview of the development

Component	Spec	Proof
preliminaries	121	93
natural deduction + lindenbaum	183	418
completeness	219	585
constructive completeness	81	258
cut-elimination + decidability IEL	193	398
cut-elimination + decidability K	116	362
$\sum$	720	2307
permutation-based cut for K	125	644
permutation-based cut for IEL	176	1045
permutation library and solver	106	143
$\sum$	407	1832
Overall $\sum$	1127	4139

Figure: Overview of the mechanization components

Future work

- Revisit constructive completeness proof
- Fiorino (2021) proposed refutation calculii and tableau system for IEL
- Investigate other semantics (e.g. Beth / topological models)

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IEL and empirical propositions

- There are verificationist theories of truth (e.g. Dummet's semantic-antitrealism)
- Account of knowledge (using BHK) will vastly differ
  - Edgington (1981): Disjunctions, blue and green vs. bleen
  - De (2013): Need different negation for empirical statements

In some sense we already know K

- K can be interpreted as propositional truncation.
- Can prove soundness of this embedding
- Rogozin (2021) suggests that IELs modality is weaker than propositional truncation

Percivals critcism of intuitionistic principles

However Percival points out that, if this solution is endorsed, it forces us to accept:

- $\neg A \leftrightarrow \neg \mathsf{K}\, A$ (falsehood of A and ignorance of A coinciding)
- $\neg(\neg \mathsf{K}\, A \wedge \neg \mathsf{K}\, (\neg A))$ (no proposition being forever undecided)

IEL response

Percivals argument rests on a classical reading of intuitionistic negation. (De Vidi and Solomon, 2001; Artemov and Protopopescu, 2016).

Can turn this argument around De (2013).

Paradox of idealization

- Apodictic numbers, primality test $=>$ knowledge in WVER is idealized
- Thus there is a P s.t. $\mathsf{K}_a(P) \supset I(a)$
- Now the proposition $P \wedge \neg\exists.I(x)$ can not be known.

Calculus for proof search

$$\frac{p_i \in \Gamma}{\Gamma \Rightarrow p_i} \qquad \frac{\bot \in \Gamma}{\Gamma \Rightarrow S} \qquad \frac{F, \Gamma \Rightarrow G}{\Gamma \Rightarrow F \supset G} \qquad \frac{F \supset G \in \Gamma \qquad \Gamma \Rightarrow F}{\Gamma \Rightarrow G}$$

$$\frac{F \wedge G \in \Gamma \qquad F, G, \Gamma \Rightarrow H}{\Gamma \Rightarrow H} \qquad \frac{\Gamma \Rightarrow F \qquad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \wedge G}$$

$$\frac{F \vee G \in \Gamma \qquad F, \Gamma \Rightarrow H \qquad G, \Gamma \Rightarrow H}{\Gamma \Rightarrow H} \qquad \frac{\Gamma \Rightarrow F_i}{\Gamma \Rightarrow F_1 \vee F_2}$$

$$\frac{\Gamma, \mathbf{K}^{-}(\Gamma) \Rightarrow F}{\Gamma \Rightarrow \mathsf{K}\, F}$$