A Completeness Proof for Intuitionistic Epistemic Logic in Coq

Initial Bachelor Seminar Talk

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Epistemic Logic

- Logics used to reason about knowledge or belief
- Most widespread used epistemic logics are (propositional) modal logics
- Idea: Just as $\Box$ and $\Diamond$ are used for necessity or possibility use $\mathbf{K}$ for knowledge
- Propositional knowledge

In a classical world

For example, Hillary Clinton did not win the 2016 US Presidential election. Consequently, nobody knows that Hillary Clinton won the election. (SEP - The Analysis of Knowledge)

$$\text{Knowledge} \Rightarrow \text{Truth}$$

It is unreasonable to claim [...] that if the number of tennis balls in my garden on 4 July 1990 is even then someone will discover that it is; the most that can be said is that it could in principle be discovered if someone bothered to look. [Williamson, 1992]

$$\text{Truth} \not\Rightarrow \text{Knowledge}$$

In an intuitionistic world

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- Truth is provability
- **Intuitionistic knowledge is the result of verification** that does not necessarily need to produce a proof [Artemov and Protopopescu, 2016]
- A proof of $\mathbf{K}\,A$ is conclusive evidence (*a certificate*) that A has a proof
- Intuitionistic truth yields intuitionistic knowledge ($A \to \mathbf{K}\,A$)
- Reject $\mathbf{K}\,A \to A$
- Relationship between Knowledge and Truth seems flipped!

Examples illustrating reading of $\mathbf{K}\,A$

A proof of $\mathbf{K}\,A$ is a certificate that A has a proof.

- Testimony from an authority
- Zero-knowledge proof
- Classified sources
- Existential generalization
- Highly probable truth
- Empirical Knowledge
- (In)formal proofs

Existential generalization (Saarbahn)

Your wallet is stolen on the Saarbahn but you cannot identify the thief.

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Your wallet is stolen on the Saarbahn but you cannot identify the thief. You have conclusive evidence that your pocket was picked, therefore

$$\mathbf{K}\,(\exists x : T(x))$$

is true. But you can't provide the witness to constructively prove $\exists x : T(x)$.

The Truth condition & Sandwich

Everything known is true

vs.

A proposition can't be known and false.

1. $\mathbf{K}\,A \to A$
2. $\neg A \to \neg \mathbf{K}\,A$
3. $\neg(\neg A \wedge \mathbf{K}\,A)$
4. $\mathbf{K}\,A \to \neg\neg A$
5. $\neg \mathbf{K}\,\bot$

2-5 are intuitionistically equivalent when $A \to \mathbf{K}\,A$ is present!

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Intuitionistic truth $\implies$ Intuitionistic Knowledge $\implies$ Classical Truth

$$A \to \mathbf{K}\, A \to \neg\neg A$$

Deduction system

Formulas are generated by the following grammar:

$$s, t \ni \mathcal{F} \coloneqq p_i \mid s \to t \mid s \wedge t \mid s \vee t \mid \mathbf{K}\, s \mid \bot \qquad (i \in \mathbb{N})$$

Taking these principles into account we use the following rules $\vdash : (\mathcal{F} \to \mathbb{P}) \to \mathcal{F} \to \mathbb{P}$:

$$\text{CTX}\ \frac{A \in \Gamma}{\Gamma \vdash A} \qquad \text{II}\ \frac{A, s \vdash t}{A \vdash s \to t} \qquad \text{IE}\ \frac{\Gamma \vdash s \qquad \Gamma \vdash s \to t}{\Gamma \vdash t}$$

$$\dots$$

$$\text{KR}\ \frac{\Gamma \vdash A}{\Gamma \vdash \mathbf{K}\, A} \qquad \text{KD}\ \frac{\Gamma \vdash \mathbf{K}\,(s \to t)}{\Gamma \vdash \mathbf{K}\, s \to \mathbf{K}\, t} \qquad \text{KT}\ \frac{\Gamma \vdash \mathbf{K}\, A}{\Gamma \vdash \neg\neg A}$$

IEL $\coloneqq$ Logic of intuitionistic knowledge (with KT)

IEL^{-} $\coloneqq$ Logic of intuitionistic belief (without KT)

Kripke Models for IEL, IEL^-

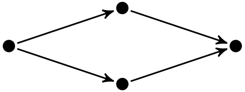
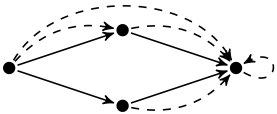


Figure: Model $\mathcal{M} = (\mathcal{W}, R, E, \mathcal{V})$

- $u \vDash \mathbf{K}\, A :\Leftrightarrow v \vDash A$ for all $v \in E(u)$
- $E \subseteq R$
- $R \circ E \subseteq E$ (shrink)
- **IEL:** $E(w) \neq \emptyset$

Kripke Models for $\mathsf{IEL}, \mathsf{IEL}^-$



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Mechanized Results

1. Soundness
2. Completeness
3. Disjunction Property, Weak Disjunction for verifications ($\mathbf{K}\,(A \lor B) \to \mathbf{K}\,A \lor \mathbf{K}\,B$ admissible)
4. Admissibility of $\mathbf{K}\,A \to A$

Completeness

- Classical proof given by Artemov
- Use a canonical model construction for both IEL, IEL^-.
- Worlds are consistent prime (i.e. $A \vdash \varphi \vee \psi \implies A \vdash \varphi$ or $A \vdash \psi$) theories (closed under $\vdash$).

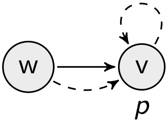
Lemma (Lindenbaum Lemma)

Any set A s.t. $A \nvdash \bot$ can be extended to a consistent prime theory.

- Lindenbaum lemma needs definiteness of $\vdash$ (i.e. $(A \vdash \phi) \vee (A \nvdash \phi)$)
- Define canonical model(s) $\mathcal{M}_{\mathbf{C}} \coloneqq (\mathcal{C}, \subseteq, \subseteq_{\mathbf{K}}, \in)$ $(\Gamma \subseteq_{\mathbf{K}} \Gamma' :\Leftrightarrow \{\phi \mid \mathbf{K}\,\phi \in \Gamma\} \subseteq \Gamma')$.

IEL $\nvdash$ **K** $A \to A$

Consider the following model $\mathcal{M}_1$:



Clearly $w \vDash \mathbf{K}\, p$ and $w \nvDash p$ therefore $w \nvDash \mathbf{K}\, p \to p$.

Admissibility $\vdash \mathbf{K}\,A \implies \vdash A$ for $\mathsf{IEL}, \mathsf{IEL}^-$

Proof.

By contraposition. Suppose $\nvdash A$. By completeness there is a model $\mathcal{M}$ and a world w s.t. $\mathcal{M}, w \nVdash A$. We can construct a new model $\mathcal{M}'$ by adding a new world w_n s.t. $w_n E w$ and $w_n R w$. Now $\mathcal{M}', w_n \nvDash \mathbf{K}\,A$. So $\mathbf{K}\,A$ is not provable. □

Embedding

- In $\mathbb{T}$: $X \to \|X\|$ but $\|X\| \nrightarrow X$
- Inhabitedness hides the computational meaning ($\|\cdot\| : \mathbb{T} \to \mathbb{P}$).

Definition

Let $\mathcal{E} : \mathbb{N} \to \mathbb{T}$ be an environment , define a function $f : \mathcal{F} \to \mathbb{T}$ by

$$\begin{aligned} f(p_i) &:= \mathcal{E}(i) \\ f(\phi \wedge \psi) &:= f(\phi) * f(\psi) \\ f(\phi \vee \psi) &:= f(\phi) + f(\psi) \\ f(\mathbf{K}\,\phi) &:= \text{inhabited}(\phi) \end{aligned}$$

Lemma

$\vdash \phi \to f(\phi)$

(Future) Results

- Soundness
- Completeness (constructive up to definiteness of $\vdash$)
- Embeddings (into Coqs Logic, IPC : only shown sound)
- Admissibility results (positive results use contraposition)
- Intuitionistic Common Knowledge [Jäger and Marti, 2016]
- (Decidability)

- Intuitionistic truth $\Rightarrow$ intuitionistic knowledge $\Rightarrow$ classical truth
- $\mathbf{K}\,A \to A$ is a **classical** principle

Thank you!

- Does intuitionistic knowledge extend to non-mathematical propositions?
- Does IEL capture all relevant aspects of an intuitionistic conception of knowledge?
- ...

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Towards Decidability (in Coq)

IEL is known to be PSPACE-complete [Krupski and Yatmanov, 2016].
Well-known approaches:

- Finite model property
- Reduction to guarded monadic fragment of FOL [1]
- Backtracking in cut-free Sequent Calculus
- Models built from finite sets of formulae

Constructive decider could be used to obtain fully constructive completeness proof.

[1] IEL satisfies conditions placed on the relations in [Alechina and Shkatov, 2006]

Embeddings II

- Into $\mathbb{P}$ embedding is certainly not complete
- Main feature of intuitionistic knowledge is preserved in $\mathbb{T}$
- [Brogi, 2020] suggest that a similiar embedding is not complete with regard to the belief interpretation (IEL^{-})
- Embedding into IPC (suggested in [Tarau, 2019]) : $\mathcal{E}(\mathbf{K}\,\phi) := (\mathcal{E}(\phi) \to E) \to \mathcal{E}(\phi)$ suggested by , where E is a fresh propositional variable (Eureka)