Saarland University
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Bachelor's thesis

Intuitionistic Epistemic Logic in Coq

Author: Christian Albert Hagemeier
Supervisor: Prof. Gert Smolka
Advisors: Dominik Kirst, Prof. Holger Sturm
Reviewers: Prof. Gert Smolka, Prof. Holger Sturm

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Abstract

Customarily, the relationship between knowledge and truth seems to be settled: Only truths can be known, therefore knowledge entails truth. This so-called truth condition is the most seldom contested principle about knowledge and is therefore also reflected in most epistemic logics, classical or intuitionistic: The principle $\mathsf{K}\, A \supset A$ (read as "if A is known, it is true") is endorsed, while $A \supset \mathsf{K}\, A$ is rejected as it seems unreasonable to conclude that an agent has knowledge of a proposition just because it is true.

In their seminal paper "Intuitionistic epistemic logic", Artemov and Protopopescu (2016) propose that a truly intuitionistic account of knowledge should endorse $A \supset \mathsf{K}\, A$ (read as "if A is proven it is known"), relying on the constructive reading of truth as provability. Nevertheless, this does not amount to rejecting the usual truth condition ($\mathsf{K}\, A \supset A$) completely, as other classically but not intuitionistically, equivalent alternatives (e.g. $\mathsf{K}\, A \supset \neg\neg A$) to the classical truth condition can be adopted.

While there is flourishing literature on IEL, so far no attempts have been made to analyze results about IEL in a constructive setting. Elaborating on the classical completeness proof, we present a constructive proof of strong quasi-completeness for IEL, which we have mechanized in the Coq proof assistant. The proof utilizes slightly modified semantics, this comes at the cost that soundness is not constructive. Our second result is a constructive decidability proof using a proof search in a cut-free sequent calculus. We generalize our method of proving cut-elimination and decidability to the classical modal logic K. We believe that this method of mechanizing cut-elimination proofs is applicable to an even larger class of modal logics. With the decidability result we obtain a constructive proof of completeness for IEL and IEL^{-}.

Lastly, we discuss two well-known epistemic paradoxes and their connection to IEL. Here we focus on the Church-Fitch paradox of knowability (Fitch, 1963) and Florio's and Murzi's paradox of idealization (Florio & Murzi, 2009) .

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Contents

1 Introduction

In this thesis, we consider two modal logics and investigate results about them in a constructive mechanized setting. Modal logics can accommodate reasoning about different modes of truth. For example, plain predicate logic is not well suited to reason about alethic (*e.g.* necessity, possibility) or deontic (*e.g.* obligatoriness, permissibility) modalities. This is mainly due to such modifiers opening intentional contexts while predicate logic is extensional. Another possible view is that modal logic is about studying non-truth functional operators. From a formal perspective, modal logics add operators to a non-modal logic which can then be used to represent the non truth-functional modality.

One special kind of modal logics are epistemic (deriving from the greek *epistēmē* meaning knowledge) logics, logics which can be used to reason about knowledge. Thus the modal operator is interpreted as an individual's or collective's knowledge and is often represented by a K. For example if A is a proposition asserting that all bachelors are unmarried, $\mathsf{K}\,A$ represents the proposition that an agent knows that all bachelors are unmarried. Similarly $\mathsf{K}\,\mathsf{K}\,A$ represents higher-order knowledge. It is obvious that knowledge modalities are not truth-functional - otherwise one would either have to know all true statements or be ignorant of all true statements. In this thesis, we investigate two logics proposed by Artemov and Protopopescu (2016). They introduce the logics IEL (intuitionistic epistemic logic), the logic of intuitionistic knowledge, and IEL$^-$, the logic of intuitionistic belief, to model an intuitionistic account of knowledge. The next chapter serves as a more detailed but informal introduction into their epistemic logic, while the formal investigation begins with Chapter 4.

Lastly, we try to establish these results in a constructive and mechanized setting. In a constructive setting not all classically valid laws are valid; at the heart is the rejection of the law of excluded middle. This principle $\forall A.\, A \lor \neg A$ is rejected, since for accepting it the constructivist would need to have an omniscient procedure which generates a proof of either A or $\neg A$ for arbitrary statements A. Note that of course rejecting the law of excluded middle does not amount to rejecting every instance of it.

We work in a mechanized setting, this basically means that all of the proofs presented in this thesis have been checked by an interactive theorem prover such as the Coq proof assistant we're working with. A mechanized proof is in some sense stricter than a non-mechanized proof, since proofs normally only need to have enough level of detail to convince an interested mathematician of the correctness. For example the completeness proof for IEL is given as a two-page appendix to Artemov and Protopopescu's paper but we need about 600 lines of code to formalize this proof.

Mechanization has allowed to find mistakes in (published) mathematical proofs. For example when Tews (2013) formalized a proof of cut-elimination for the class of co-algebraic logics in Coq, he found slight mistakes or gaps in the reasoning of the paper by Pattinson and Schröder (2010). Reasoning in a proof assistant based on constructive logic like Coq also allows for a fine-grained analysis how strong classical axioms (if they are needed) need to be. In this chapter we present the outline and our main contributions.

1.1 Outline

From a high-level perspective this thesis consists of two parts: One informal part (Chapters 2, 7) discussing the philosophical motivation for IEL and two related epistemic paradoxes. The remaining chapters (namely chapters 3, 4, 5, 6) provide a formal investigation into the logic and are framed in-between the two non-mechanized chapters. The parts are independent so a reader interested in the technical results may want to skim or skip Chapter 2.

The second chapter of this thesis is concerned with an introduction to IEL and IEL^{-}. Here we introduce the philosophical motivation behind IEL and introduce their Kripke-style semantics in an informal way.

The third chapter lists some preliminaries and quickly introduces the constructive meta-theory the proofs are formalized in.

The fourth chapter, is the start of our formal investigations into IEL. After introducing the syntax of IEL, we formally introduce models. We then discuss a classical completeness proof, elaborating on the one given by Artemov and Protopopescu (2016).

The fifth chapter presents a modified version of the completeness proof which was given in the preceding chapter. The construction utilizes a slightly modified forcing relation which allows us to prove strong quasi-completeness and (presupposing decidability) completeness constructively. This comes at the price that soundness cannot be proven without relying on the constructively invalid law of excluded middle.

Chapter 6 concludes our formal investigation into IEL. We prove, basing on previous work by Dang (2015), Krupski and Yatmanov (2016), and Smolka and Brown (2012), the decidability of IEL. We obtain this by implementing a proof search in a cut-free sequent calculus for IEL, which has been proven equivalent to the natural deduction system using a cut-elimination proof, and proving the algorithms' correctness.

The penultimate chapter further develops the philosophical investigation into IEL. We first present the Church-Fitch paradox and afterwards discuss Artemov and Protopescu's solution to it, which is based on using IEL as the underlying logic. Secondly, we investigate in how far an intuitionistic conception of knowledge is threatened by Florio and Murzi's paradox of idealization. If their argument is forceful, it commits intuitionists to the existence of an unverifiable truth, which would in some sense refute IEL's co-reflection principle. Furthermore we discuss arguments against IEL principles, for example by Percival (1990).

In the concluding chapter we discuss related and further work. That chapter also includes an overview of the accompanying Coq development.

1.2 Contributions

To summarize, our technical contributions include mechanizations of

- a cut-elimination proof for the intuitionistic modal logics IEL and IEL^{-} and the classical modal logic K in the Coq proof assistant,
- a decidability proof for IEL and IEL^{-} and the classical modal logic K,
- a mechanization of the well-known classical completeness proof (*i.e.* $\mathcal{T} \Vdash A \to \mathcal{T} \vdash A$) using the law of excluded middle,
- a quasi strong-completeness proof for IEL and IEL^{-} (*i.e.* $\mathcal{T} \Vdash' A \to \neg\neg\mathcal{T} \vdash A$), and
- a proof of completeness for IEL and IEL^{-} (*i.e.* $\Gamma \Vdash' A \to \Gamma \vdash A$) and some consequences.

Our non-technical contributions include an introduction into IEL and its principles and a discussion of two epistemic paradoxes and their relationship to IEL.

Most results in this thesis have been formalized in the Coq proof assistant. Such results have been linked with its respective counterpart in the formalization and clicking on the theorems title will open the formalization. Formalized theorems are recognizable by a visual cue.

The accompanying coq development and documentation can be found on this bachelor projects' webpage.[1]

[1] http://www.ps.uni-saarland.de/~hagemeier/bachelor.php

2
A Primer on Intuitionistic Epistemic Logic

In most theories of knowledge the relationship between truth and knowledge is simple: What is known is true, but just because something is true it does not need to be known. For example, the truth component is explicitly mentioned in the well-known tripartite analysis of knowledge as justified, true belief. This is also reflected in most epistemic logics, i.e. logics with a knowledge modality K, which take $\mathsf{K}\,A \supset A$ to be a defining principle, expressing the facticity of knowledge. However, Artemov and Protopopescu (2016) propose that a faithful account of knowledge understood intuitionistically reverses this relationship. Accordingly Artemov and Protopopescu's intuitionistic epistemic logic (IEL) validates not the reflection, but the so-called co-reflection[1] axiom $A \supset \mathsf{K}\,A$. In this chapter, we will first motivate this logic, then discuss consequences concerning the notion of proof that arise from adopting the co-reflection principle and lastly give an informal introduction to IELs Kripke-semantics. We will discuss how knowledge is analyzed in Kripke models and how this interacts with the interpretation of Kripke-models as modelling information states of an idealized reasoner or mathematician.

[1] In the context of Fitch's paradox (Fitch, 1963) this principle is also known as strong verificationism (this naming was introduced in Williamson (2000)).

2.1 Intuitionistic Epistemic Logic

Intuitionism views mathematics as a web of constructive proof patterns and matching definitions of objects (van Benthem, 2009). Let us emphasize two consequences of this view: Intuitionists assign a different meaning to the logical constants than classical mathematicians. For classical mathematicians, the meaning of a logical constant is given by a truth table, while intuitionists identify them with their contribution to the proof conditions of sentences. The logical constants are infused with a proof-theoretic spirit, summed up in the Brouwer-Heyting-Kolmogorov (BHK) interpretation (van Benthem, 2009). For example, a proof of $A \wedge B$ is a pair containing a proof of A and a proof of B, and a proof of an implication $A \supset B$ is a construction transforming proofs of A into proofs of B. Another difference is that proofs are mental constructions and a proposition is only seen as true once a proof has been constructed. Thus truth is identified with provability.

Artemov and Protopopescu (2016) claim that an intuitionist conception of knowledge is verification-based. That is, an intuitionist can know a proposition whenever she is certain (can verify) that the proposition has been constructively proven. However she does not need to have actually constructed the proof, she only needs to be certain that a constructive proof exists.[2] We will call such a piece of verifiable evidence that a constructive proof exists a *certificate*.

[2] We will deal with the question if the proof must exist or if possibility of proof suffices later.

Theoretically speaking there are numerous possible relationships between proofs and certificates. Artemov and Protopopescu argue that every proof is a certificate but that there are non-proof certificates (*i.e.* they are in a strict subset relation). That means that from any constructive proof we need to be able to obtain a certificate and that there are other methods of obtaining certificates. If there were no other sources of certificates besides constructive proofs, knowledge and truth would coincide. We will consider examples of non-proof verifications shortly, but let us first argue why every (intuitionistic) proof is a certificate: It is a defining property of a proof, that it can be checked, and of course, the proof itself establishes that a proof exists. Since proofs are used to convince readers of the truth of a statement, if the reader cannot verify the proof, it has no value. Formally Artemov and Protopopescu adopt the following axioms for their intuitionistic epistemic logic (IEL):

Definition 2.1 *The axioms of IEL are those of intuitionistic propositional calculus and*

- $A \supset \mathsf{K}\,A$ *(co-reflection)*
- $\mathsf{K}\,(A \supset B) \supset \mathsf{K}\,A \supset \mathsf{K}\,B$ *(distribution)*
- $\mathsf{K}\,A \supset \neg\neg A$ *(intuitionistic reflection).*

While distribution is a standard and accepted axiom,[3] the other axioms look strange from a classical perspective. The intuitionistic reflection principle can be seen to express the idea that the certificates need not yield constructive proofs but guarantee the existence of a (classical) proof. Read intuitionistically, it guarantees that no proof of $\neg A$ exists, when A is known. Using the well-known double-negation translation for propositional logic (known as Glivenko's theorem, (Glivenko, 1929)) it sandwiches intuitionistic knowledge between intuitionistic and classical truth. The logic without the intuitionistic reflection axiom is called IEL$^-$, the logic of intuitionistic belief.

[3] Of course, distribution is controversial too, *e.g.* it gives rise to logical omniscience (see Rapaport & Vardi, 1988).

Is the proposed strict-subset relationship between certificates and proofs reflected in the logic? One might have the idea that K can be defined in terms of classical truth. This would equate IEL with the logic HDN $\Box$ from Došen (1984), which proves $\mathsf{K}\,A \leftrightarrow \neg\neg A$. However, this would (by Glivenkos theorem) imply, that every theorem of classical logic is known. However, Artemov and Protopopescu argue that $\neg\neg A \supset \mathsf{K}\,A$ should not be endorsed in an epistemic logic of knowledge, since then the agent would know $p \vee \neg p$ for any proposition since $p \vee \neg p$ is a classical tautology and thus $\neg\neg(p \vee \neg p)$ is a theorem of intuitionistic propositional logic (by Glivenko's theorem). Fortunately, the logics are proven to be different (Artemov & Protopopescu, 2016), therefore intuitionistic knowledge and double negated truth do not coincide.[4]

We will now return to the existence of non-proof verifications as this ties in well with the co-reflection principle. If all verifications were proofs, it would make no sense to only allow the (intuitionistically) weaker inference to $\neg\neg A$ from $\mathsf{K}\, A$. So, how is the co-reflection principle motivated?

We give two examples (from Artemov and Protopopescu (2016)) for certificates obtainable from other sources than constructive proofs. As a first example, we outline how knowledge can be obtained from zero-knowledge proofs.

Consider a sudoku puzzle and two players Alice and Bob. Bob knows a solution and wants to convince Alice of that fact without revealing the solution. Using a zero-knowledge protocol,[5] Alice can verify that Bob is in possession of a solution, this allows her to prove $\mathsf{K}\,(\exists x.\, S(x))$ (we use the unary predicate $S(x)$ to denote that x is a solution of the sudoku puzzle). Thus she would have obtained a solution to the puzzle, in contrast to the idea that the solution shall not be reconstructable from a zero-knowledge protocol. Note, that the crucial part here is not, that Bob obtained a solution (and thus too has a proof of $\exists x.S(x)$) but only that Alice is sure that a constructive proof exists. Assume for example, there was a computer program that could tell if a sudoku is solvable without revealing the solution or constructing one itself, even if Alice were to run this program on the sudoku and the program would attest to Alice that the sudoku is solvable, without revealing the solution, she could assert $\mathsf{K}\,(\exists x.\, S(x))$ (this is of course assuming the computer program can be verified and there are no hardware-errors).

There are variants of this example, e.g. knowledge from classified sources or knowledge from authority (Artemov & Protopopescu, 2016). For example, most mathematicians claim to *know* Fermat's Last Theorem and can even use it in proofs (this can be seen as an instance of the distribution axiom) but can probably not present a proof when asked to produce one. While these examples make the intuitionistic reflection principle plausible, the co-reflection principle still looks puzzling. Read classically it states omniscience (e.g. the agent knows every truth) and seems to flip the (classical) relationship between truth and knowledge: Everything true is known and knowledge does not yield (constructive) truth.

But under an intuitionistic reading, it seems more plausible, as expressed by Bell and Hart (1979):[6]

> Read the proof; thereby you come to know that the sentence is true. Reflecting on your recent learning, you recognize that the sentence is now known by you; this shows that the truth is known.

One possible objections against this could be that a reasoner could come to possess a proof which super-seeds her intellectual capacities. In this case she could read the proof but obtain no knowledge, since she does not understand the proof. However I do not believe that this is a valid criticism. The scheme $A \supset \mathsf{K}\, A$ read intuitionistically only

[4] It is possible to show that $\neg\neg(p \vee \neg p) \supset \mathsf{K}\, p \vee \neg p$ is not admissible by constructing a countermodel, that is a model in which $\neg\neg(p \vee \neg p)$ is satisfied but $\mathsf{K}\,(p \vee \neg p)$ is not.

[5] such protocols exists, for instance see Gradwohl et al. (2009)

[6] However Bell and Hart seem to find this line of argument unsatisfying, since we *surely have good inductive grounds for believing that there are truths as yet unknown.*

expresses that once an agent has come to recognize a (constructive) proof of A, she immediately knows that A is true. A does not become known as soon as she is presented with a proof but only as soon as she has realized that what she has just read is a proof of A.

However the main issue here is what is meant by *proof*, the argument crucially depends on the proof being available to the agent. We will elaborate on this in Section 2.2.

One interesting result concerning this understanding of knowledge is its similarity to type inhabitation (also known as truncated types or squash types) in constructive type theory (the similarity is noted by Artemov and Protopopescu). The inhabitance type $\|A\|$ of a type A only captures that A has an inhabitant, but the inhabitant cannot be extracted from it. In this sense, it is quite similar to certificates, which guarantee that a proof exists but offer no way of obtaining the proof. Interestingly, in Coq's Type Theory, we can interpret K as inhabitedness and the IEL axioms hold in this interpretation. In particular, from $\|A\|$ it is possible to obtain a proof of $\neg\neg A$.

There are actually multiple equivalent intuitionistic alternatives to the classical truth condition. The following theorems are all equivalent in the presence of co-reflection and can serve as the intuitionistic truth condition on knowledge:

- $\neg(A \wedge \neg\mathsf{K}\,A)$
- $\neg\mathsf{K}\,\bot$
- $\neg A \supset \neg\mathsf{K}\,A$
- $\neg\neg(\mathsf{K}\,A \supset A)$

Especially the fourth formulation makes it obvious that intuitionistic reflection is nothing else than the double-negated classical truth-condition on knowledge.

2.2 *Notion of proof*

The crucial question is what exactly constitutes an intuitionistic proof. In this section, we will analyze different conceptions of intuitionistic proofs and check if those are compatible with the IEL axioms.

The issue we focus on here is if proofs are types or tokens. We will consider Williamson's (1988) idea to model proofs as types and his argument against the validity of co-reflection in his intuitionistic epistemic logic. Then we will discuss some difficulties from identifying truth with having constructed a proof token. Most of the arguments presented in this section can also be found in Murzi (2010), however regarding the proof as types proposal I come to a different conclusion. While Murzi concludes that committing to co-reflection necessitates rejecting the "proofs as types"-proposal, I believe that by using a broader interpretation of proof-token similarity this proposal can work well with the co-reflection principle.

Proofs as types Williamson defines $\mathsf{K}\,A$ to hold if there exists a time t s.t. A has been proven at t. His idea is to define proof types in terms of structural identity of proof tokens. He introduces proof types as an *ontologically neutral* (Williamson, 1988) concept: Two proof tokens are of the same proof-type if and only if they have the same structure and conclusion, but may occur at different times (we shall denote similarity of proof tokens by ~). Proof types under this conception are nothing more than proof tokens grouped together by similarity (like equivalence classes). For example, two structurally identical proofs of the Pythagorean theorem carried out at different times would count as different proof tokens of the same type, while two proofs of the Pythagorean theorem with a different structure would count both as different proof types and tokens. There is no such thing as a general proof type of the Pythagorean theorem (Murzi, 2010) , unless all proofs of the Pythagorean theorem were identical in structure.

Talk of proof types can always be reduced to talk of similar proof tokens. This also applies to the BHK-semantics, as can be seen by considering the case of implication. The usual explanation is that a proof of $A \to B$ is a function (in the sense of a procedure or construction) transforming proofs of A into proofs of B. So if proofs are types, a function mapping proof types to proof types, how can this be reduced to proof tokens? Williamson (1988) suggests interpreting the conditional as a **unitype** function between proof tokens that is a function which preserves similarity of tokens *i.e.* if π and ρ are similar, so are $f(\pi)$ and $f(\rho)$. So formally, a function f between proof tokens is unitype iff

$$\forall \pi\rho.\ \pi \sim \rho \implies f(\pi) \sim f(\rho).$$

So if co-reflection were valid, there would have to be a unitype function mapping proof tokens of A to proof tokens of $\mathsf{K}\,A$. His argument against the co-reflection principle is now that no such function f can exist. To see this, consider a proposition p which is currently undecided. That is, no proof of p but also no proof of $\neg p$ has yet been constructed. However since co-reflection is valid, the function must exist. Now Williamson's trick is to take two similar hypothetical proof tokens of p, π_1 and π_2 where π_1 is constructed on a Monday and π_2 is constructed on a Tuesday (for them to be similar they only need to have the same proof structure).

Williamson now claims that $f(\pi_1) \nsim f(\pi_2)$, since they were proved on different days and the day of the proof is an essential part of the proof of $\mathsf{K}\,A$, by the definition of the semantics for K.

The notion of hypothetical proof tokens might seem strange (and maybe even unnecessary) at first. If the functions mapping proof-tokens from A to $\mathsf{K}\,A$ would only need to operate on real proof tokens, it would be possible to map every proof token of A (even those constructed at different times) to the same proof token of $\mathsf{K}\,A$ (namely the one which was constructed first), thus the function would be unitype since for all proofs π, π_2 of A, $f(\pi) = f(\pi_2)$ would hold and thus $f(\pi) \sim f(\pi_2)$. But if the proposition has not been decided (*i.e.* there is no proof of A nor one of $\neg A$) this strategy is not possible.

An objection by Martino and Usberti (1994) is that the function need not operate on hypothetical but only on real proof tokens, since hypothetical proof tokens do not exist. However, as Murzi (2010) points out, intuitionists routinely assert conditionals without knowing whether the antecedent is true. Consider proving $\neg p$, which means proving $p \to \bot$. If the function which proves this, would only need to work on real proof tokens, it would need to only map real proof tokens of p to real proof tokens of $\bot$. So if $\neg p$ holds, the function has nothing to map, since there are no proof tokens of $\bot$ there can not be any of p. Operating on hypothetical proof tokens is in some sense also reflected in natural deduction and other calculi: When proving a conditional $\Gamma \vdash s \supset t$, after applying the introduction rule for implication, one has to prove $s, \Gamma \vdash t$, but here too, we work with the hypothetical assumption that s is true regardless if we have actually obtained or even can obtain a proof of s.

However the more troubling argument (which phrased slightly differently can be found in van Atten (2018), Artemov and Protopopescu (2016) and Usberti (2016)) is that there seems to be no clear motivation why similarity of proof-tokens needs to be defined in such a restricted way. One possible way out is not to mandate that the time at which a proposition was proven plays any role in the proof structure of $\mathsf{K}\,A$. That is, by using an interpretation of K that does not include a temporal component, just like in IEL, the objection no longer works.

A similar strategy is to broaden the notion of similarity. If proof tokens are defined to be similar if they have the same conclusion, that is, the internals of the proof are of no concern (as in the Curry-Howard isomorphism), there seems to be no reason why there cannot be a unitype function mapping proof-tokens of A to proof-tokens of $\mathsf{K}\,A$.

A second possible criticism of Williamson's approach has to do with proof types of statements for which no proof has been produced (yet) or for which no proof token might ever be constructed. For example there could be statements which are too long to ever be verified. If proof types are nothing else than tokens grouped by similarity this proof type might not exist, since no proof token of it will ever be constructed. One possible solution here would be to commit to a platonic realm of (hypothetical) proof tokens, however this destroys the idea, that proof types are ontologically neutral and nothing more than (existing) proof tokens grouped by similarity. As Murzi (2010) points out, it might be objected that the notion of a platonist proof is an inherently realist one. For example Dummett (1982, p. 90) notes this:

> A platonist will admit that, for a given statement, there may be neither a proof nor a disproof of it to be found; but there is no intelligible anti-realist notion of truth for mathematical statements under which a statement is true only if there is a proof of it, but may be true because such a proof exists, even though we do not know it, shall never know it, and have no effective means of discovering it.

Dummets main argument now is that if we commit to an objective (platonic) realm of proofs, there is no reason to not commit to a parallel conception of mathematical objects or a platonist conception of natural numbers.

Cozzo remarks that a platonic realm of proofs is "very different from a realistically conceived transcendent reality consisting of objects which are conceptually completely independent of our cognitive practices." (Cozzo, 1994, p. 75). Since the realist will still assert that statements without a proof can be true, while the platonistic intuitionist is only committed to the existence of a realm of proofs which she can recognize (the proofs are not independent from our cognitive practices) understanding. For example, we know the assertibility conditions of Goldbach's conjecture, thus even a platonic realm of proofs (or proof tokens) is in someway linked with our understanding.

A similar stance for example can be found in the Coq community (or more broadly the interactive theorem proving community with constructive background). Here only constructing a proof token (e.g. a coq script which will correspond to a term proving the proposition) is seen as evidence that one has proven the statement – but many members would see a statement as true regardless if the proof has been constructed. So while truth is identified with the type having a member, only constructing and having access to a token can make us aware of this. In this sense it seems as if Coq users are tacitly committed to the existence of a platonic realm of proofs (which is still more graspable than a platonic realm of ungraspable truths).

In summary while under Williamson's notion of similarity the co-reflection principle is faced with problems, a more general notion of proof type seems to validate the co-reflection principle, but seems to presuppose the existence of a platonic realm of proofs. If one does not assume a platonic realm of proofs, there seem to be issues with statements for which no proof token will ever be constructed.

Proofs as tokens If truth is identified with the actual possession of a proof token, there are other counterintuitive consequences, since this leads to truth having a temporal component. Consider a valid inference from a set of sentences Γ to a conclusion p. Assume that a proof token for each sentence in Γ but not for p exists. If validity requires preservation of truth, the inference would be wrong, so it would turn correct as soon as a proof token of p had been found. Another well-known problem is with past-tensed decidable statements for which all evidence has been lost.

Of course, platonist or timeless conceptions of truth are in theory available to the intuitionist - intuitionism isn't per se committed to identifying truth with the actual (or possible) construction of a proof. However such conceptions of proof seem not to enjoy co-reflection, since just from the abstract existence of proof, it is impossible to say anything about the construction of a proof of $\mathsf{K}\,A$. Before a proof of A can be transformed into a proof of $\mathsf{K}\,A$ it needs to be constructed at least once by an actual agent.

Another distinction which can be made is if truth is viewed in an actualist or possibilist fashion, *i.e.* if it suffices if the proof were constructible or if it actually needs to be constructed. IEL seems not to be committed to either of these, co-reflection is certainly compatible with an actualist view. Even on a possibilist view, if it was possible to construct a proof of A it would be possible to construct a proof of $\mathsf{K}\,A$, thus in principle both notions are possible. For a broader discussion on actualist or possibilist notions of truth we refer the reader to Raatikainen (2004).

2.3 Models

This section will not give a formal introduction into the models used for IEL but instead try to convey an intuition, how (intuitionistic) knowledge fits into Kripke-model theoretic terms. Kripke models have applications in intuitionistic as well as classical logic. In classical logics Kripke models are used to model intensional operators such as necessity, possibility or knowledge.

We will first introduce how knowledge is interpreted in terms of Kripke models. Kripke models consist of a set of possible worlds and interpret truth of formulas relative to a world w. For this they additionally have an accessibility relation on the worlds (which might be subject to some constraints) and a valuation function assigning truth values to atomic sentences at specific worlds. In the classical case for example $A \supset B$ is forced at world w a model M, in formulas $\mathcal{M}, w \Vdash A \supset B$ if either $\neg A$ or B is forced at w in M.

The common approach to modeling knowledge using Kripke models is to use the *indistinguishability* interpretation of knowledge (Rendsvig & Symons, 2019). Consider an agent walking the sunny streets of San Francisco who knows nothing about the weather in London.[7] Therefore, she only considers worlds possible where it is sunny in San Francisco, but she would consider both worlds possible where it is sunny or where it is raining in London (more precisely, she considers any weather in London possible). On the other hand she has no knowledge about the weather in London. Thus she considers both situations possible where it is raining in London and situations where it is sunny in London. Therefore she neither knows that it is sunny nor that it is raining in London.

[7] This example is taken from Fagin et al. (2004).

This motivates defining in terms of possible worlds: An agent knows a proposition if it holds in all worlds she considers possible. Thus she knows something, if all worlds she considers possible are indistinguishable with respect to the proposition in question. If she had some means to exclude the possibility that it is raining in London (for example by listening to a weather broadcast), she would know that it is not raining in London.

In an intuitionistic setting, the possible worlds are often thought to represent possible states of information of an ideal reasoner.[8] This ideal reasoner also called *creating subject* by Brouwer constructs all mathematical objects and statement. In any situation there are multiple possible ways the ideal reasoner could extend her knowledge. All of these are different states in the model. The partial ordering of these

[8] This exposition is based on similar expositions in (Artemov & Protopopescu, 2016; van Dalen, 1986).

states represents the evolutionary process of the information states, which are only increasing (there is no revision of beliefs or truths for an ideal reasoner, she can only construct new objects and thus learn new truths) (Proietti, 2012). We denote this partial order by $\leq$ and call it cognition relation.

Artemov and Protopopescu (2016) suggest to incorporate K into this picture by including an additional epistemic access relation. In IEL the successor states which are relevant for the K-modality can be thought of as an audit set, a set of worlds in which verifications can occur. Hence knowledge (and belief in IEL^{-}) are modeled as truth in any audit world. There are two additional conditions placed on the relationship between the audit sets and cognition relation.

Firstly, the audit sets are monotone with respect to the cognition relation. Thus if $v \leq w$ (w is cognitively-accessible from v), then w's audit set needs to be fully contained in v's. This corresponds to the Kripkean ideology that as the ideal reasoner creates more object he can only become more confident and "things become more certain in the process of discovery" (Artemov & Protopopescu, 2016) .

Secondly, any epistemic successor must also be a cognitive successor. This condition corresponds to the idea that if it is possible to do a verification in a specific state, it is a possible scenario.

These conditions lead to validity of the co-reflection principle. For IEL, there is an additional condition, which blocks $\mathsf{K}\bot$ from being semantically entailed at any world. In chapter 4 we will present a mechanized soundness and (classical) strong completeness proof for these models.

3
Technical Preliminaries

The ambient metatheory we use in this thesis is the Calculus of Inductive Constructions (Coquand & Huet, 1988; Paulin-Mohring, 1993), which is the foundation of the Coq Proof Assistant (The Coq Development Team, 2020). The calculus of inductive constructions distinguishes two kinds of universes. One impredicative universe $\mathbb{P}$ of propositions and an infinite hierarchy of predicative universes $\mathbb{T}_0 \subseteq \mathbb{T}_1 \subseteq \dots$. We usually drop the index of the type level and just use $\mathbb{T}$ to refer to the universe of types. With some notable exceptions, it is not possible to eliminate from propositions into types. For example it is not possible to extract a witness from a proof of an existential statement.

Readers who are only familiar with set theoretic notations shall not be discouraged: most parts of the lemma statements and proofs, in the level of detail presented in the thesis, can probably also be read as if we were working in a constructive metatheory based on set-theory. However the differences are of course apparent in the mechanization.

3.1 Numbers

Types are introduced by inductive definitions. For example the type of natural numbers can be defined in the following way:

$$\mathbb{N} : \mathbb{T} \coloneqq 0 \mid S\,n \quad (n : \mathbb{N})$$

This definition establishes that elements of the type $\mathbb{N}$, *i.e.* natural numbers, are constructed from 0 and the successor function $S : \mathbb{N} \to \mathbb{N}$. For example, the number 4 is represented as $S\,S\,S\,S\,0$.

Addition and subtraction are defined by (well-founded) recursion, the definitions are:

$$\begin{aligned} 0 + x &= x & \qquad S(x) - S(y) &= x - y \\ Sx + y &= S(x + y) & z - y &= z \end{aligned}$$

We will use $n + 1$ as notation for $S(n)$.[1] Similarly we write $n - k$ for the truncating subtraction (*i.e.* $2 - 3 = 0$).

[1] Strictly speaking, the terms $S\,n$ and $n + 1$ are not structurally equal, since addition $+ : \mathbb{N} \to \mathbb{N} \to \mathbb{N}$ is typically defined by recursion on the first argument.

3.2 Lists

Lists play an essential role in our development, since we will use those to represent finite sets of formulas. Lists over a type T are either the empty list [] or the concatenation of any list L of type T with an element t of type T, denoted by $t :: L$. In the context of derivation systems (*e.g.* natural deduction systems or sequent calculi) we will also use the common notation L, t.[2] We define membership $\in: T \to \mathcal{L}(T) \to \mathbb{P}$. The length of a list is denoted by $\|A\|$. Two lists are sublists of each other, denoted by $A \subseteq B$, if and only if $\forall x. x \in A \to x \in B$. We define two notions of equality on lists, set equivalence and multiset equivalence, which is the same as two lists being permutations of each other.

[2] In those contexts the order of the elements will not matter.

Definition 3.1 *Two lists are equivalent, denoted by $A \equiv B$ if and only if every element of A is contained in B and vice versa.*

$$A \equiv B :\Leftrightarrow A \subseteq B \wedge B \subseteq A$$

With $A \equiv_P B$ we will denote that two lists are permutations of each other. Formally permutations over lists of type A are defined using an inductive predicate $\equiv_P$: $L(A) \to \mathcal{L}(A) \to \mathbb{P}$, with the following constructors:

$$\frac{}{[\,] \equiv_P [\,]} \qquad \frac{L_1 \equiv_P L_2}{x :: L_1 \equiv_P x :: L_2} \qquad \frac{L_1 \equiv_P L_2}{x :: y :: L_1 \equiv_P y :: x :: L_2}$$

$$\frac{L_1 \equiv_P L_2 \qquad L_2 \equiv_P L_3}{L_1 \equiv_P L_3}$$

The first constructor establishes that empty lists are permutations of one another. The second allows to append a single argument to any established permutation. The third allows to swap the first two arguments and the fourth expresses transitivity of the permutation relation. Permutations are

3.3 Sets and Decidability

While sets are not primitive to type theory, sets can be represented as predicates, *i.e.* functions taking an element of a type yielding a proposition. For example, the set of numbers divisble by 4 can be represented by a predicate $P : \mathbb{N} \to \mathbb{P}$, which is defined by $P\,n := \exists n. 4 * m = n$. We will use the usual mathematical notation $x \in P$ instead of $P\,x$. Last let us introduce the notion of decidability of a predicate. Consider any predicate $P : T_1 \to \ldots \to T_n \to \mathbb{P}$. We call it *decidable* if we can construct a function $f : T_1 \to \ldots \to T_n \to \mathbb{B}$, such that $\forall t_1 : T_1 \ldots, \forall t_n : T_n. p\, t_1 \ldots t_n = \text{true} \leftrightarrow P\, t_1 \ldots t_n$.

4
Syntax and Semantics of IEL

In the last chapter concepts like possible world semantics and the syntax of IEL where only informally introduced. In this chapter we will formally introduce the syntax and semantics of IEL and present the classical strong completeness proof for IEL (as given in Artemov & Protopopescu, 2016). The presented proof is an extension of the well-known Henkin-style (Henkin, 1949) (strong) completeness proof using a canonical model construction for intuitionistic logic with Kripke semantics.

4.1 Syntax

We start by defining the type of formulas.

Definition 4.1 (Syntax of Intuitionistic propositional logic) *The formulas of IEL are the elements of the following inductive data type:*

$$A, B : \mathcal{F} \coloneqq p_i \mid \bot \mid A \vee B \mid A \wedge B \mid A \supset B \mid \mathsf{K}\, A \quad (i \in \mathbb{N})$$

As earlier, we will use the $\supset$ symbol to denote the (material) conditional.

We define the size of a formula using a straightforward inductive definition.

Definition 4.2 (Formula size) *The function* size : $\mathcal{F} \to \mathbb{N}$ *is recursively defined by*

$$\begin{aligned}
\mathrm{size}(A \circ B) &\coloneqq \mathrm{size}(A) + \mathrm{size}(B) + 1 \\
\mathrm{size}(\mathsf{K}\, A) &\coloneqq \mathrm{size}(A) + 1 \\
\mathrm{size}(\bot) &\coloneqq 0 \\
\mathrm{size}(p_i) &\coloneqq 0
\end{aligned}$$

In the above definition $\circ$ is used as a placeholder for any binary operator.

We will later prove statements by induction on the size of formulas.

We will in some proofs need to enumerate all the formulas, for this we have to show that $\mathcal{F}$ is countable that is there exists a function $d : \mathbb{N} \to \mathcal{F}$ which is surjective.

Lemma 4.3 *The type $\mathcal{F}$ is countable.*

Proof. This fact can be established by standard techniques, e.g. injecting into any countable type. We prove this fact by injecting into a general tree type. □

We will denote the i-th formula of the enumeration by F_i. Next we will look at a formal proof calculus for IEL (and of course IEL$^-$).

4.2 Natural Deduction

We will use a natural deduction system, which is easily derived from the Hilbert system introduced in Artemov and Protopopescu (2016). Natural Deduction formulations for IEL are also presented in Krupski and Yatmanov (2016) or Rogozin (2021). Natural deduction (originally developed by Gentzen (1935b) and Jaśkowski (1934)) formalizes the idea to draw a single inference from a set of assumptions, so formally the provability relation relates sets of assumptions with a conclusion.[1]

However there is the issue of how to represent the set of assumptions in a proof assistant. In mathematical reasoning we are used to reasoning with infinite sets of assumptions, this would motivate defining the *contexts* as predicates $\mathcal{T} : \mathcal{F} \to \mathbb{P}$, with the implicit convention that $A \in \mathcal{T}$ if and only if $\mathcal{T}\,A$ holds.

However dealing with finite contexts, which can be represented as lists of formulas, is easier in the proof assistant. We choose the best of both worlds: We define the entailment predicate $\vdash$ on lists of formulas, that is $\vdash: \mathcal{L}(\mathcal{F}) \to \mathcal{F} \to \mathbb{P}$ and reduce reasoning about (possibly) infinite contexts to deduction in finite sublists. This reduction does not change the set of formulas derivable from infinite contexts, since any derivation must be of a finite length and can therefore also only use a finite number of assumptions. We call finite sets of formulas contexts and (possibly) infinite sets of formulas theories.

[1] Natural deduction is called natural since in comparison with axiomatic systems common at the time of its invention it more closely resembles the natural / pretheoretically accepted style of reasoning (Pelletier & Hazen, 2012).

Definition 4.4 (Natural Deduction for IEL and IEL$^-$) *We define natural deduction for IEL as a predicate $\vdash: \mathcal{L}(\mathcal{F}) \to \mathcal{F} \to \mathbb{P}$.*

$$\frac{A \in \Gamma}{\Gamma \vdash A}\,\text{A} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}\,\text{E}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\,\text{II} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \supset B}{\Gamma \vdash B}\,\text{IE}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\,\text{DIL} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\,\text{DIR} \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C \quad \Gamma \vdash A \vee B}{\Gamma \vdash C}\,\text{DE}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,\text{CI} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\,\text{CEL} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\,\text{CER}$$

$$\frac{\Gamma \vdash \mathsf{K}\,(A \supset B)}{\Gamma \vdash \mathsf{K}\,A \supset \mathsf{K}\,B}\,\text{KIMP} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash \mathsf{K}\,A}\,\text{COREFL} \qquad \frac{\Gamma \vdash \mathsf{K}\,A}{\Gamma \vdash \neg\neg A}\,\text{IREFL}$$

A natural deduction system for IEL$^-$ is obtained from the IEL system by removing the intuitionistic reflection rule (IREFL). Note, that the natural deduction system for IEL is an extension of the natural deduction system for intuitionistic propositional logic by the rules for the K-operator (distribution, co-reflection and intuitionistic reflection).

Definition 4.5 (Entailment in theories) *Let $\mathcal{T} : \mathcal{F} \to \mathbb{P}$ be a (possibly) infinite context. We define derivability by*

$$\mathcal{T} \vdash A :\iff \exists L^{\mathcal{L}(\mathcal{F})}. L \subseteq \mathcal{T} \wedge L \vdash A$$

That is, an infinite context $\mathcal{T}$ derives A if there exists a list L which only contains elements from $\mathcal{T}$ s.t. $L \vdash A$. We implicitly use lists as theories in the above notation. We will adopt the following conventions: We will use uppercase greek letters e.g. Γ, Δ to denote finite contexts (e.g. lists of formulas) and calligraphic uppercase letters e.g. $\mathcal{T}$ to denote possibly infinite contexts (theories) in the predicate-representation.

We can now proceed to show some structural properties of the natural deduction calculus.

Lemma 4.6 (Theory weakening) *If $\mathcal{T}_1 \vdash A$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$ then $\mathcal{T}_2 \vdash A$.*

Proof. Assume $\mathcal{T}_1 \vdash A$, therefore there is a list $\Gamma \subseteq \mathcal{T}_1$ s.t. $\Gamma \vdash A$. We can now use the same Γ to instantiate the existential allowing us to prove $\mathcal{T}_2 \vdash A$. □

Proving weakening for contexts needs an induction.

Lemma 4.7 (Weakening) *If $\Gamma \vdash A$ and $\Gamma \subseteq \Omega$ then $\Omega \vdash A$.*

Proof. Induction on the derivation $\Gamma \vdash A$ with Ω quantified. □

Once we have proven weakening for the list-based natural deduction, we are able to show all the natural deduction rules admissible for theories. For this, we need weakening, since we need to concatenate lists in cases where more than 1 premise is used in the natural deduction rules.

An important property (that is elementary in natural deduction calculi) is the deduction theorem (and it's reverse).

Lemma 4.8 (Implication agreement) *$A, \mathcal{T} \vdash B$ if and only if $\mathcal{T} \vdash A \supset B$.*

Proof. The left to right direction is just implication introduction. The other direction is proven using implication elimination and weakening. □

Truth conditions As mentioned in Chapter 2 there are multiple equivalent intuitionistic truth-conditions. We do not verify here that the logics resulting in adding each to the natural deduction system for IEL$^-$ are the same, but only verify that we can derive the other (intuitionistic) truth conditions when adopting $\mathsf{K}\,A \supset \neg\neg A$.

Lemma 4.9 *In IEL, $\neg\mathsf{K}\,\bot$.*

Lemma 4.10 *In IEL for every A, $\neg(\mathsf{K}\,A \wedge \neg A)$.*

Lemma 4.11 *In IEL for every A, $\neg A \supset \neg\mathsf{K}\,A$.*

Lemma 4.12 *In IEL for every A, $\neg\neg(\mathsf{K}\,A \supset A)$.*

4.3 Kripke Models

We will now introduce the semantics of IEL in a formal manner. As discussed in the last chapter, we will use possible world semantics, namely Kripke models to represent the semantics. We start by defining Kripke models for IEL and IEL$^-$.

Definition 4.13 *A* Kripke model *for* IEL$^-$ *is a quadruple* $(\mathcal{W}, \mathcal{V}, \leq, \leq_{\mathbf{K}})$ *consisting of*

- *a type of* worlds $\mathcal{W}$,
- *a* valuation *function* $\mathcal{V} : \mathcal{W} \to \mathbb{N} \to \mathbb{P}$ *which maps worlds and propositional variables to propositions,*
- *a cognition relation* $\leq : \mathcal{W} \to \mathcal{W} \to \mathbb{P}$*, which can be any preorder on* $\mathcal{W}$,
- *a verification relation* $\leq_{\mathbf{K}}$.

Additionally models have to fulfill the following constraints on the relations:

1. *the valuation is monotone w.r.t* $\leq$ *i.e. if* $w \in \mathcal{W}$ *is an arbitrary world and* $\mathcal{V}(w,i)$ *holds* $\mathcal{V}(w',i)$ *must hold at any* $\leq$*-successor* w'
2. *the verification relation is contained in the cognition relation, i.e.* $w \leq_{\mathbf{K}} u$ *implies* $u \leq v$ *for every* $u, w \in \mathcal{W}$.
3. $\leq \circ \leq_{\mathbf{K}} \quad \subseteq \quad \leq_{\mathbf{K}}$*, that is if* $u \leq v$ *and* $v \leq_{\mathbf{K}} w$ *then* $u \leq_{\mathbf{K}} w$ *for every* $u, v, w \in \mathcal{W}$.

In IEL models additionally there cannot be any blind worlds[2] *, formally* $\forall w. \exists u. w \leq_{\mathbf{K}} u$.

[2] Blind worlds in this context are worlds without any $\leq_{\mathbf{K}}$-successor

The condition that no world in an IEL-model has an empty set of $\leq_{\mathbf{K}}$–successors will guarantee that $\mathsf{K}\perp$ will not be forced at any world. Next we define the forcing relation $\Vdash$ for semantic entailment. We use the common notation $\mathcal{M}, w \Vdash A$ to denote that the formula A is forced in a model $\mathcal{M}$ at a world w. We sometimes leave the model implicit.

Definition 4.14 (Kripke forcing relation) *Let* $\mathcal{M} := (\mathcal{W}, \mathcal{V}, \leq, \leq_{\mathbf{K}})$ *be an IEL (or IEL-) model. We define the forcing relation by induction on the formula:*

$$\mathcal{M}, w \Vdash p_i :\Leftrightarrow \mathcal{V}(w,i)$$
$$\mathcal{M}, w \Vdash A \wedge B :\Leftrightarrow \mathcal{M}, w \Vdash A \wedge \mathcal{M}, w \Vdash B$$
$$\mathcal{M}, w \Vdash A \supset B :\Leftrightarrow \forall w'. w \leq w' \to \mathcal{M}, w \Vdash A \to \mathcal{M}, w \Vdash B$$
$$\mathcal{M}, w \Vdash A \vee B :\Leftrightarrow \mathcal{M}, w \Vdash A \vee \mathcal{M}, w \Vdash B$$
$$\mathcal{M}, w \Vdash \mathbf{K} A :\Leftrightarrow \forall w', w \leq_{\mathbf{K}} w' \to \mathcal{M}, w \Vdash A$$

Definition 4.15 (Forcing with theories) *Let* $\mathcal{T}$ *be a theory. We write* $\mathcal{T} \Vdash A$ *to express that in any model and world where every formula of* $\mathcal{T}$ *is a semantic consequence,* A *is too.*

$$\mathcal{T} \Vdash A :\Leftrightarrow \forall \mathcal{M} \forall w. (\forall B \in \Gamma. \mathcal{M} \Vdash_w B) \Rightarrow \mathcal{M} \Vdash_w A$$

By identifying contexts with theories, we can use the same notation for contexts.

Of course we now want to relate the deduction and the semantics. The usual way to do this, is to prove both soundness and completeness. Soundness establishes that any derivable formula is valid in all models and completeness establishes the converse: if a formula is valid in all models, it can be derived. We start by proving soundness. Inorder to prove this we first need to show the monotonicity of forcing with respect to the cognition relation.

Lemma 4.16 (Monotonicity of Kripke forcing) *For every w and u, $w \Vdash A$ and $w \leq u$ imply $u \Vdash A$.*

Proof. Straightforward induction on the formula. □

With monotonicity we can establish soundness.

Lemma 4.17 (Soundness) $\Gamma \vdash A \implies \Gamma \Vdash A$

Proof. The proof is by induction on $\Gamma \vdash A$.

Case ndA: Assume $\Gamma \vdash A$ was derived using the assumption rule, thus $A \in \Gamma$. We have to show $\Gamma \Vdash A$. So let w be an arbitrary world in a model $\mathcal{M}$ where every formula of Γ is a semantic consequence. Thus $\mathcal{M}, w \Vdash A$, since $A \in \Gamma$.

Case ndII: Assume $\Gamma \vdash A_1 \supset A_2$ was derived using the implication introduction rule. So let w be an arbitrary world in a model $\mathcal{M}$ where every formula of Γ is a semantic consequence. We need to show $\mathcal{M}, w \Vdash A_1 \supset A_2$. Therefore let w' be any $\leq$-successor of w with $\mathcal{M}, w' \Vdash A_1$. We need to show $\mathcal{M}, w' \Vdash A_2$, for this we apply the inductive hypothesis and need to show that every formula in Γ, A_1 is a semantic consequence at w'. For A_1 this is our assumption, for any formula in Γ we can use monotonicity and our assumption that every formula of Γ is a semantic consequence at w.

Case CoRefl: Assume $\Gamma \vdash \mathsf{K}\, A$ was derived using the co-reflection rule, therefore we have $\Gamma \vDash A$ by the inductive hypothesis. We need to show that A is a semantic consequence at any $\leq_{\mathbf{K}}$–successor. This holds by monotonicity, since every $\leq_{\mathbf{K}}$-successor is a $\leq$-successor too.

Note, that proving soundness for contexts will can be used to establish soundness for theories. □

Lemma 4.18 (Soundness for theories) $\mathcal{T} \vdash A \implies \mathcal{T} \Vdash A$

Proof. Assume $\mathcal{T} \vdash A$. By definition of entailment in theories, there is a context Γ s.t. $\Gamma \vdash A$ and $\Gamma \subseteq \mathcal{T}$. By soundness for contexts, any model in which all members of Γ are forced, forces A too. Since any model forcing $\mathcal{T}$ must force Γ we are done. □

4.4 Classical Completeness

In this section we will elaborate the classical strong completeness proof for IEL and IEL- which uses a canonical model construction. Since the proof is classical, we use the law of excluded middle

$$\forall p^{\mathbb{P}}.p \vee \neg p \tag{LEM}$$

multiple times, as it for reasoning classically e.g. it allows for proofs by contradiction.[3]

[3] $\forall p^{\mathbb{P}}.\neg\neg p \supset p$ is equivalent to LEM.

4.4.1 *Lindenbaum Extension*

We will show that any set of formulas $\mathcal{T}$ (in the abstract possibly infinite sense) can be extended into a set of formulas enjoying desirable properties. We can then use those sets as worlds in the canonical model. Let us first define the properties formally.

Definition 4.19 (Theory Properties) *Let $\mathcal{T}$ be a theory. We call $\mathcal{T}$*

- prime, *if for any $A \vee B \in \mathcal{T}$, either $A \in \mathcal{T}$ or $B \in \mathcal{T}$*
- deductively-closed, *if any derivable formula is already contained i.e. $\mathcal{T} \vdash A \implies A \in T$*
- consistent *if $\mathcal{T} \nvdash \bot$*

A theory with all three above properties is called saturated.

We can now state the Lindenbaum lemma.

Lemma 4.20 (Lindenbaum) *Let $\mathcal{T}$ be a theory and A a formula such that $\mathcal{T} \nvdash A$. Then there is a saturated theory $\mathcal{T}'$ extending $\mathcal{T}$ and $\mathcal{T}' \nvdash A$.*

Proof. We will only sketch the proof here, a full formal proof of a similar lemma will be given in the chapter on the constructive strong quasi-completeness (Chapter 5) proof. The basic idea is to greedily extend the context $\mathcal{T}$ by any formula which does not cause the extension to derive A. For this, we can use the enumeration of the formulas (e.g. of the type $\mathcal{F}$) $F_1, F_2, \ldots,$. We construct a chain of theories $\mathcal{T}_0, \mathcal{T}_1, \ldots,$ by setting

$$\mathcal{T}_n := \begin{cases} \mathcal{T} & n = 0 \\ T_n, F_n & \text{if } \mathcal{T}_n, F_n \nvdash A \\ \mathcal{T}_n & \text{otherwise} \end{cases}$$

Note, that while this looks like a classical definition (and arguably is one), it can be represented as

$$\mathcal{T}_{n+1} := \lambda F.F \in \mathcal{T}_n \vee (\mathcal{T}_n, F_n \nvdash A \wedge F = F_n)$$

without having to do an explicit case analysis using LEM (thus the definition is constructive). We claim, that $\mathcal{T}_\infty := \bigcup_{n \in \mathbb{N}} \mathcal{T}_n$ is a theory which has the desired properties.[4] A crucial property of this chain is, that any derivation from the full union $\mathcal{T}_\infty$ can be lowered to a single level, to put it formally $\mathcal{T}_\infty \vdash B \iff \exists n.\mathcal{T}_n \vdash B$. By induction it is possible to show that $\forall i.\mathcal{T}_i \nvdash A$. This also implies consistency. Both the theory and primeness property can be proven by deriving a contradiction. To establish primeness we need to use LEM. So let

[4] This union is represented as $\lambda x.\exists n.\mathcal{T}_n\, x$.

$B \vee C \in \mathcal{T}_\infty$. By LEM $\mathcal{T}_\infty \vdash B \vee \mathcal{T}_\infty \nvdash B$ and $\mathcal{T}_\infty \vdash C \vee \mathcal{T}_\infty \nvdash C$. Doing a case analysis, the cases where either B or C can be derived are easy (using the theory property of $\mathcal{T}_\infty$). If $\mathcal{T}_\infty \nvdash B$ and $\mathcal{T}_\infty \nvdash C$ it is possible to show $\mathcal{T}_\infty \vdash B \supset A$ and $\mathcal{T}_\infty \vdash C \supset A$. Now by disjunction elimination $\mathcal{T}_\infty \vdash A$ holds which allows us to derive a contradiction. □

4.4.2 Canonical Model

We can now prove completeness employing a canonical model construction (as presented in Artemov & Protopopescu, 2016) . Canonical models of a logic $\mathcal{L}$ are models which refute any non-theorem of a logic. This can be used to prove the contrapositive of completeness (and thus classically prove completeness) i.e. $\nvdash A \implies \nVdash A$, since the canonical model and the respective world where the non-theorem A is refuted are enough to show that A is not a semantic consequence in every model and every world.

The worlds in the canonical model will be built from saturated theories. An important property is the *truth lemma*: Membership in the saturated theories which are worlds in the canonical model and semantic entailment agree.

Let us first define the canonical model. In this section we will abuse notation and use greek letters for theories too.

Definition 4.21 (Canonical Model) *The canonical model for IEL is defined as $\mathcal{M}_\mathbf{C} = (\mathcal{W}_\mathbf{C}, \mathcal{V}_\mathbf{C}, <, <_\mathbf{K})$ with*

- *the type of all saturated theories $\mathcal{W}_\mathbf{C}$,*
- $\Gamma < \Omega :\Leftrightarrow \Gamma \subseteq \Omega$
- $\Gamma <_\mathbf{K} \Omega :\Leftrightarrow \Gamma_\mathbf{K} \subseteq \Omega$, *here $\Gamma_\mathbf{K}$ denotes the downward* **K***-projection i.e.* $\Gamma_\mathbf{K} \coloneqq \{A \mid \mathsf{K}\, A \in \Gamma\}$.
- *the valuation relation defined by $\mathcal{V}(p_i, \Gamma) \coloneqq p_i \in \Gamma$.*

In a similar fashion the canonical model $\mathcal{M}^-_\mathbf{C} = (\mathcal{W}^-_\mathbf{C}, \mathcal{V}^-_\mathbf{C} <^-, <^-_K)$ for IEL$^-$ *can be defined. $\mathcal{W}^-_\mathbf{C}$ is the type of all saturated* IEL$^-$ *theories.*

The main difference between the canonical model for IEL and IEL$^-$ is that the underlying notion of theories changes, since the ND-system is different. For example $\neg\mathbf{K}\bot$ is a theorem at any canonical IEL world, but there is a world in the canonical model for IEL$^-$ where it will not be a semantic consequence. Let us now show, that the constructed models satisfy all the constraints on models. For proving that the canonical model for IEL is a model we need an additional lemma, which we call the modal shift lemma.

Lemma 4.22 (Modal Shift Lemma) $\Gamma_\mathbf{K} \vdash A \implies \Gamma \vdash \mathsf{K}\, A$.

Proof. Assume $\Gamma_\mathbf{K} \vdash A$ thus there are assumptions $A_1, \ldots, A_m$ s.t. $\{A_1, \ldots, A_n\} \vdash A$. By K-necessitation and some modal reasoning $\mathsf{K}\, A_1 \wedge \mathsf{K}\, A_2 \wedge \cdots \wedge \mathsf{K}\, A_m \vdash \mathsf{K}\bot$. Since $\mathsf{K}\, A_1, \ldots, \mathsf{K}\, A_m \in \Gamma$, we obtain $\Gamma \vdash \mathsf{K}\, A$.

Lemma 4.23 *The canonical models are models for* IEL, IEL$^-$.

Proof. We only prove the IEL-version, since it is a generalization of the IEL$^-$ version. It is trivial to see that $<$ is a preorder (it inherits this property due to the fact that $\subseteq$ is a preorder). Let us now check that $\Gamma <_{\mathbf{K}} \Omega \implies \Gamma < \Omega$. So assume $\Gamma_1 \subseteq \Omega$ and let $A \in \Gamma$ be arbitrary. Since the worlds in the canonical model are deductively-closed and $A \supset \mathsf{K}\, A$ is valid in both IEL$^-$ and IEL, $\mathsf{K}\, A \in \Gamma$. From the assumption $\Gamma <_{\mathbf{K}} \Omega$ we now obtain $A \in \Omega$.

Secondly, we need to check that $\Gamma \overset{(i)}{<} \Omega \overset{(ii)}{<_{\mathbf{K}}} \Theta$ implies that $\Gamma <_{\mathbf{K}} \Omega$. So let $\mathsf{K}\, A \in \Gamma$ be arbitrary, by (i) we know $\mathsf{K}\, A \in \Omega$ and by (ii) we obtain $A \in \Theta$, thus $\Gamma <_{\mathbf{K}} \Theta$.

Last we need to check, that in the canonical model for IEL no worlds are without a $\leq_{\mathbf{K}}$-successor. We claim that $\Gamma_{\mathbf{K}}$ can be Lindenbaum-extended to a consistent prime theory, thus every world has a K-successor in the canonical IEL-model. For this we only need to show, that at any world Γ it is the case that $\Gamma_{\mathbf{K}} \nvdash \bot$. Since Γ is a consistent IEL world, $\Gamma \nVdash \mathsf{K} \bot$ (this is one of the possible intuitionistic truth conditions, Lemma 4.9) , now by the modal shift lemma, $\Gamma_{\mathbf{K}} \nvdash \bot$ thus we can use the Lindenbaum lemma to extend it to a world in the model, which obviously is a $<_{\mathbf{K}}$-successor. □

We can now prove the truth lemma, establishing the connection between membership in saturated theories and forcing in the respective worlds of the canonical model, again we only prove the IEL-version, but the proof for IEL$^-$ is the same.

Lemma 4.24 (Truth lemma) *For any world* $\Gamma \in \mathcal{W}_{\mathbf{C}}$ *and formula* A

$$\Gamma \Vdash A \iff A \in \Gamma.$$

Proof. The proof is by induction on A. We only prove selected cases here.

Case p_i*:* Assume $\Gamma \Vdash p_i$ by definition of $\Vdash$ this is the case if and only if $\mathcal{V}(w, i)$. But that is the case if and only if $p_i \in \Gamma$.

Case $\mathbf{K}\, A$*:* For the only-if direction assume $\mathsf{K}\, A \in \Gamma$, thus for any Ω with $\Gamma <_{\mathbf{K}} \Omega$ we have $A \in \Omega$. Applying the inductive hypothesis yields $\Omega \Vdash A$. Since this is true for any $<_{\mathbf{K}}$-successor we have proven $\Gamma \Vdash \mathsf{K}\, A$. For the other direction, assume (by contraposition) $\mathsf{K}\, A \notin \Gamma$ but $\Gamma \vdash \mathsf{K}\, A$. By the modal shift lemma $\Gamma_{\mathbb{K}} \nvdash A$,thus by the Lindenbaum lemma, it can be extended to a saturated theory (thus a world in the model) which does not derive A. But this world is a $<_{\mathbf{K}}$-successor to Γ but does not force A - a contradiction.

Case $A \vee B$*:* Assume $\Gamma \Vdash A \vee B$, thus $\Gamma \Vdash A \vee \Gamma \Vdash B$ (by the definition of $\Vdash$). Applying the inductive hypothesis we obtain $\Gamma \vdash A \vee \Gamma \vdash B$. Now by using disjunction introduction and the fact that Γ is a theory, $A \vee B \in \Gamma$ can be proven.

For the converse assume $A \vee B \in \Gamma$. Since Γ is prime either $A \in \Gamma$ or $B \in \Gamma$. Assume w.l.o.g. $A \in \Gamma$, by the inductive hypothesis $\Gamma \Vdash A$ but that suffices to prove $\Gamma \Vdash A \vee B$. □

Theorem 4.25 (Strong completeness) $\mathcal{T} \Vdash A \implies \mathcal{T} \vdash A$

Proof. We prove the contraposition. Therefore assume $\mathcal{T} \nvdash A$. By the Lindenbaum lemma the theory $\mathcal{T}$ can be extended to a saturated theory $\mathcal{T}'$ s.t. $\mathcal{T}' \nvdash A$. But now by the truth lemma $\mathcal{M}_{\mathbf{C}}, \mathcal{T}' \nVdash A$ - a contradiction. □

The presented strong completeness proof used classical principles at multiple steps, namely once in the Lindenbaum lemma, multiple times in the truth lemma (contraposition was used) and one-top contraposition. In the next chapter we will show how some of these can be weakened.

4.5 Further Results

In the following section we take a look at some selected results which use the results we just proved. While the reflection rule is not derivable in IEL (and if it was derivable, it would collapse knowledge and truth), we can show it admissible. Apart from the reduction between IEL and IEL$^-$ derivability, all of these results have already been presented by Artemov and Protopopescu.

Lemma 4.26 (Admissibility of reflection) *The reflection rule is admisssible,* i.e. $\vdash \mathsf{K}\, A$ *implies* $\vdash A$.

Proof. We prove the contraposition, thus it suffices to construct a countermodel to $\mathsf{K}\, A$ from a countermodel for A. So assume $(\mathcal{M} := (\mathcal{W}, \mathcal{V}, \leq, \leq_{\mathbf{K}})$ is a countermodel to A i.e. there is a world $w_A \in \mathcal{W}$ s.t. $\mathcal{M}, w_A \nVdash A$. We construct a new model by adding an additional world, with a single **K**-successor w_A. At that world $\mathsf{K}\, A$ is not a semantic consequence. □

We can now show that IEL has the disjunction property.

Lemma 4.27 (Disjunction Property) $\vdash A \vee B \implies (\vdash A) \vee (\vdash B)$

Proof. We show the contrapositive. Therefore assume $\neg(\vdash A \vee \vdash B)$. Now classically $\nvdash A$ and $\nvdash B$. By strong completeness a models $\mathcal{M}_A = (\mathcal{W}_A, \mathcal{V}_A, \leq_A, \leq_{\mathbf{K},A})$ and $\mathcal{M}_B = (\mathcal{W}_B, \mathcal{V}_B, \leq_B, \leq_{\mathbf{K},B})$ and worlds $w_a \in W_A$ and $w_B \in W_B$ exist s.t. $\mathcal{M}_A, w_A \nVdash A$ and $\mathcal{M}_B, w_B \nVdash B$.

We assume that the worlds are disjoint and let w be a world not contained in either model. We create a new model by adding w as a new world. Formally we construct a model $\mathcal{M} := (W_A \cup W_B \cup \{w\}, \mathcal{V}, \leq, \leq_{\mathbf{K}})$ where $\mathcal{V}$ agrees with $\mathcal{V}_A$ and $\mathcal{V}_B$ on the respective worlds and $V(p_i, w) := \bot$. Similarly the cognition and verification relation are the union of the models' relations and additionally $w \leq w'$ and $w \leq_{\mathbf{K}}$ for any world w'. Now if $\mathcal{M}, w \Vdash A \vee B$, by definition of $\Vdash$ either $\mathcal{M}, w \vDash A$ or $\mathcal{M}, w \Vdash B$. Both cases give rise to a contradiction. □

However IEL does not enjoy the disjunction property for verifications.

Lemma 4.28 *The rule* $\mathsf{K}\,(A \vee B) \supset \mathsf{K}\, A \vee \mathsf{K}\, B$ *is not admissible.*

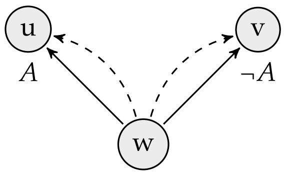

Figure 4.5.1: Countermodel to $\mathsf{K}\,(A \vee B) \supset \mathsf{K}\,A \vee \mathsf{K}\,B$

Proof. Consider the following IEL model.

Obviously $w \Vdash \mathsf{K}\,(A \vee \neg A)$ but neither $w \Vdash \mathsf{K}\,A$ nor $w \Vdash \mathsf{K}\,\neg A$ hold. $\square$

Despite Lemma 4.28 IEL and IEL$^-$ both have a weak disjunction property for verifications:

Lemma 4.29 *If $\vdash \mathsf{K}\,(A \vee B)$ then $\vdash \mathsf{K}\,A$ or $\vdash \mathsf{K}\,B$.*

Proof. If $\vdash \mathsf{K}\,(A \vee B)$. By admissibility of reflection we have $\vdash A \vee B$. With the disjunction property we have $\vdash A$ or $\vdash P$. With co-reflection we have $\vdash \mathsf{K}\,A$ or $\vdash \mathsf{K}\,B$ as desired. $\square$

One interesting result we can show is that derivability in IEL can be reduced to derivability in IEL$^-$. This is because one of the possible ways to express the truth condition is $\neg\mathsf{K}\,\bot$. Thus by adding this as an assumption to the context IEL$^-$ can simulate IEL-derivations.

Lemma 4.30 *If $\mathcal{T} \vdash A$ in IEL we can derive $\neg\mathsf{K}\,\bot, \mathcal{T} \vdash A$ in* IEL$^-$.

5
Towards Constructive Completeness

In this chapter we first investigate how to improve on the classical completeness proof presented in the Chapter 4. Our strategy is to first refine the Lindenbaum construction by constructivizing it and secondly use slightly modified Kripke semantics. Using these we can then proof strong quasi-completeness, that is double negated strong completeness, constructively. Using decidability, which we will prove in chapter 6, we can even obtain a constructive proof of completeness.

5.1 Constructive Lindenbaum

We first establish a constructive variant of the Lindenbaum lemma. Most of the properties of a saturated theory can be established constructively, however we need to modify the primeness property. For this we introduce the notion of quasi-primeness, which is simply doubly-negated primeness. This is since we only needed classical reasoning when proving the primeness in the classical Lindenbaum construction. Now using quasi-primeness, we can circumvent using full excluded middle.

Definition 5.1 (Quasi-primeness) *Let $\mathcal{T}$ be a theory. It is* quasi-prime *if and only if*

$$\forall A^{\mathcal{F}} B^{\mathcal{F}}.\, A \vee B \in \mathcal{T} \implies \neg\neg(A \in \mathcal{T} \vee B \in \mathcal{T}).$$

In this section we will present a proof in more detail than in the first chapter. We start by defining a general theory extension. Throughout this section we will fix a theory $\mathcal{T}$ and a formula $A_\perp$ which is not derivable from $\mathcal{T}$ (i.e. $\mathcal{T} \nvdash A_\perp$), which we will extend into a consistent quasi-prime theory.

Definition 5.2 (Theory extension) *Let $\mathcal{T}$ be a theory. We define the extension $\mathcal{T} \oplus B$ as*

$$\mathcal{T} \oplus B := \lambda F.\, F \in \mathcal{T} \vee (\mathcal{T}, F \nvdash A \wedge F = B).$$

That is just like in the preceding chapter we add a formula if and only if adding it does not cause the extension to derive $A_\perp$.

We again define a chain of theories $\mathcal{T}_0, T_1, \ldots$ where just as before we set

$$\mathcal{T}_0 \coloneqq \mathcal{T}$$
$$\mathcal{T}_{i+1} \coloneqq \mathcal{T}_n \oplus F_i.$$

We define the infinite union $\mathcal{T}_\infty \coloneqq \lambda F. \exists n.\, F \in \mathcal{T}_n$.

Lemma 5.3 (Theory extension) *For any theory $\mathcal{T}$ and formula B the theory $\mathcal{T}$ is fully contained in the extension: $\mathcal{T} \subseteq \mathcal{T} \oplus B$.*

Proof. Let C be an arbitrary formula, we assume $C \in \mathcal{T}$ and have to show $C \in \mathcal{T} \oplus B$ which is just $C \in \mathcal{T} \vee (B = C \wedge B, \mathcal{T} \nvdash A_\perp)$. But this is easy since our assumption is the left disjunct. □

With this we can prove some subset properties for the complete chain.

Lemma 5.4 (Chain subset properties) *We can prove the following subset properties for the chain:*

1. *$\mathcal{T} \subseteq \mathcal{T}_i$ for all i*
2. *$\mathcal{T}_i \subseteq T_j$ for all $i < j$*
3. *$\mathcal{T}_i \subseteq \mathcal{T}_\infty$ for all i.*

Proof. All of these are easily obtained from a subset property for theory extension, Lemma 5.3. (i) can be established using induction on i, (ii) by induction on $i < j$ and (iii) by instantiating the existential with i. □

Before we can establish that the extension does not derive $A_\perp$, we first prove a chain-compactness result.

Lemma 5.5 (Chain compactness) *For any formula F, a derivation in $\mathcal{T}_\infty$ can be lowered to a concrete chain-level, i.e.*

$$\forall F. \mathcal{T}_\infty \vdash F \iff \exists n. \mathcal{T}_n \vdash F.$$

Proof. The right-to-left direction is easily obtained using weakening and the chain subset properties (Lemma 5.4).

For the other direction suppose $\mathcal{T}_\infty \vdash F$. Thus there is a list L with formulas taken from $\mathcal{T}_\infty$ s.t. $L \vdash F$. Now we need to show that there is a level i in the theory chain at which F can be derived. We show this using weakening with a specific theory level, thus we need to prove $L \subseteq \mathcal{T}_i$ for a fixed i. We have $F \in L \implies \exists n. F \in \mathcal{T}_n$, we can extract the level for every item and compute their maximum m. Using the subset-properties every formula is contained in the $\mathcal{T}_m$ and thus $\mathcal{T}_m \vdash F$. □

We now show that the extension does not derive $A_\perp$ (provided $\mathcal{T} \nvdash A_\perp$).

Lemma 5.6 $\mathcal{T}_\infty \nvdash A_\perp$

Proof. By the last lemma, it suffices to show $\forall n. \mathcal{T}_n \nvdash A_\perp$. We prove this by induction on n.

Case 0: Since $\mathcal{T}_0 = \mathcal{T}$, this is our assumption.

Case $n+1$: By the inductive hypothesis we know that $\mathcal{T}_n \nvdash A_\perp$. We need to show $\mathcal{T}_{n+1} \nvdash A_\perp$. For deriving a contradiction, assume $\mathcal{T}_{n+1} \vdash A_\perp$. If we can show that $\mathcal{T}_{n+1}$ is extensionally equivalent with $\mathcal{T}_n$, we are done, since then we have both $\mathcal{T}_n \vdash A_\perp$ and $\mathcal{T}_n \nvdash A_\perp$ as hypotheses allowing us to derive a contradiction.

To show, $\mathcal{T}_{n+1} \equiv \mathcal{T}_n$, we only consider the $\subseteq$-direction, since the other one is easily obtained from the closure properties. Informally this is simple to show, since the theory is only extended, if its extension does not derive $A_\perp$, but we have already assumed that $\mathcal{T}_n \nvdash A_\perp$ by the inductive hypothesis. Formally let $A \in \mathcal{T}_{n+1}$ thus either $A \in T_n$ (so we are done) or $A = \mathcal{F}_i$ and $\mathcal{T}_i, F_i \nvdash A_\perp$. We do a proof of contradiction and thus must proof $\mathcal{T}_i, F_i \vdash A_\perp$, we can use weakening with $\mathcal{T}_{i+1}$ next, which we have as an assumption. □

We will now show that $\mathcal{T}_\infty$ is deductively closed.

Lemma 5.7 *If $\mathcal{T}_\infty \vdash B$ then $B \in \mathcal{T}_\infty$*

Proof. Assume $\mathcal{T}_\infty \vdash B$. Of course it suffices to show, that there is a n s.t. $B \in \mathcal{T}_n$ (because of the subset-properties, lemma 5.4). There is an index i s.t. $F_i = B$ because the enumeration is surjective. We will show, that $B \in \mathcal{T}_{i+1}$.[1]

Using the definition of $\mathcal{T}_{i+1}$ we either have to show $B \in \mathcal{T}_i$ or $F_i = B \wedge \mathcal{T}_i, B \nvdash A_\perp$. We choose to prove the right disjunct. Since that is a conjunction, we need to prove both conjuncts, since the first is an assumption, we only need to prove that second, i.e. $\mathcal{T}_i, B \nvdash A_\perp$. We will derive a contraction by assuming $\mathcal{T}_i, B \vdash A_\perp$ and proving $\mathcal{T}_\infty \vdash A_\perp$, contradicting lemma 5.6. We use the implication introduction rule with B and need to prove both $\mathcal{T}_\infty \vdash B \supset A_\perp$ and $\mathcal{T}_\infty \vdash B$. The second is easy, since we have $\mathcal{T}_\infty \vdash B$ as an assumption. For the first, we can use implication agreement (which in this case is just the elimination rule for implication) and need to prove $\mathcal{T}_\infty, B \vdash A_\perp$, which we can obtain from our assumption $\mathcal{T}_n, B \vdash A_\perp$ by weakening. □

[1] Recall, that the i-th formula in the enumeration is inserted into $\mathcal{T}_{i+1}$.

Last we will show the quasi primeness. We will now show, why we can reason classically when proving negated goals.

Lemma 5.8 (Prime LEM) *For any fixed proposition p, $\neg\neg(p \vee \neg p)$.*

Proof. Let p be arbitrary and assume $\neg(p \vee \neg p)$. We construct a proof of falsity by using our hypothesis and choosing to prove the right disjunct. Now we have to prove $\perp$ with p as an additional assumption. But by using $\neg(p \vee \neg p)$ once more, we again have to prove $p \vee \neg p$, but this time we choose to prove the left disjunct, since we have a proof of p as an assumption. □

Informally speaking this lemma will allow us to reason classically (but only finitely often, *i.e.* we cannot use the law of excluded middle infinitely often) whenever the goal is negated. This can be captured in the following lemma:

Lemma 5.9 *For any p and q*

$$((q \vee \neg q) \to \neg p) \to \neg p$$

Proof. Assume $(q \vee \neg q) \to \neg p$ and p. We have to derive a contradiction, by Lemma 5.8 it suffices to prove $\neg(q \vee \neg q)$. Thus we have $q \vee \neg q$ as an additional assumption and still need to derive a contradiction. Now we can deduce $\neg p$ using $q \vee \neg q$. But then we have both $\neg p$ and p as assumptions yielding the desired contradiction. □

Lemma 5.10 *If $\mathcal{T}_\infty \nvdash B$ then $\neg\neg\mathcal{T}_\infty \vdash B \supset A$.*

Proof. Assume $\mathcal{T}_\infty \nvdash B$. We know that B is contained in the enumeration , there is an i s.t. $F_i = B$. We can locally decide $B \in \mathcal{T}_{i+1} \vee B \notin \mathcal{T}_{i+1}$. In the first case we can derive a contradiction from $\mathcal{T}_\infty \nvdash B$ using weakening and the assumption rule.

The second case is more involved. We need to proof $B \in \mathcal{T}_{i+1}$. We prove the right disjunct i.e. $B = F_i \wedge \mathcal{T}_i, B \nvdash A$. As always, the first disjunct is our assumption. For the second disjunct we can first introduce it and use implication agreement to obtain a proof of $\mathcal{T}_i \vdash B \supset A$ and need to prove $\bot$. We can apply our assumption $\mathcal{T}_\infty \nvdash B \supset A$, thus we need to prove $\mathcal{T}_\infty \vdash B$. Now using weakening, we can use $\mathcal{T}_i \vdash B \supset A$ to complete the proof. □

Lemma 5.11 (Quasi-primeness) *$\mathcal{T}_\infty$ is quasi-prime.*

Proof. Since our goal is a double negated, we can do a single intro and obtain $\neg(\mathcal{T}_\infty \vdash B \vee C \implies T_\infty \vdash B \vee \mathcal{T}_\infty \vdash C)$ as an assumption. Since our goal is falsity, we can reason classically to decide $B \in \mathcal{T}_\infty \vee B \notin \mathcal{T}_\infty$ and $C \in \mathcal{T}_\infty \vee C \notin \mathcal{T}_\infty$. Just as in the classical case, the cases where one of the formulas are derived are simple. Now assume $B \notin T_\infty$ and $C \notin T_\infty$. We can assert $\neg\neg\mathcal{T}_\infty \vdash A_\bot$ by removing the double-negations and then using disjunction elimination and lemma 5.10. But this contradicts $\mathcal{T}_\infty \nvdash A_\bot$. □

As a result we obtain the constructive version of the Lindenbaum lemma.

Lemma 5.12 (Constructive Lindenbaum) *Any theory $\mathcal{T}$ which does not derive A can be extended into a consistent, quasi-prime, deductively-closed theory which does not derive A.*

Proof. An immediate consequence of above lemmata. □

5.2 Modal shifting lemma

The modal shifting lemma,showing that if $\Gamma_{\mathbf{K}} \vdash A$ then $\Gamma \vdash \mathsf{K} A$ was not really proven in the last chapter, the proof was informal - this section provides a more formal proof - however it is not essential for understanding the modified completeness proof.

We begin by defining a shift function, shifting a list of formulas into an implication chain.

Definition 5.13 (Shifting) *Define* shift : $\mathcal{L}(\mathcal{F}) \to \mathcal{F} \to \mathcal{F}$ *by*

$$\begin{aligned} \text{shift}\, \emptyset\, A &:= A \\ \text{shift}\, B, \Gamma\, A &:= B \supset (\text{shift}\, \Gamma\, A) \end{aligned}$$

We will use the notation $\Gamma \rightsquigarrow A$ for $\text{shift}\,\Gamma\,A$. Note, that there are different possibilities for the order when shifting.

We will prove some lemmas about shifting, which in the end will be used to prove the modal shifting lemma.

Lemma 5.14 (Partial Shifting) *If* $\Gamma_1, \Gamma_2 \vdash A$ *then* $\Gamma_2 \vdash \Gamma_1 \rightsquigarrow A$.

Proof. Induction on Γ_2 with A and Γ_1 quantified. □

Next we can prove one lemma about shifting and applying **K** to a list of formulas. We will write $\mathsf{K}\,\Gamma$ to denote the list obtained by applying K to every element in the list Γ (formally $\mathsf{K}\,\Gamma := \text{map}\,\mathsf{K}\,\Gamma$).

Lemma 5.15 *If* $\Gamma_1 \vdash \mathsf{K}\,(\Gamma_2 \rightsquigarrow A)$ *then* $\Gamma_1 \vdash (\mathsf{K}\,\Gamma_2) \rightsquigarrow \mathsf{K}\,A$

Proof. Induction on Γ_2 with Γ_1 quantified. □

Now we can prove the modal shifting lemma.

Lemma 5.16 (Modal shifting lemma) *If* $\mathcal{T}_\mathbf{K} \vdash A$ *then* $\mathcal{T} \vdash \mathsf{K}\,A$.

Proof. From the assumption $\mathcal{T}_\mathbf{K} \vdash A$ we know that there is a context Γ s.t. $\Gamma \vdash A$. We will use $\mathsf{K}\,\Gamma$ to instantiate the existential i.e. we have to prove $\mathsf{K}\,\Gamma \vdash \mathsf{K}\,A$ (and also that every element of $\mathsf{K}\,\Gamma$ is an element of $\mathcal{T}$, which is easy). We can use the identity $\mathsf{K}\,\Gamma = \mathsf{K}\,\Gamma +\!\!+ []$ and can now use lemma 5.14 and need to prove $\vdash (\mathsf{K}\,\Gamma) \rightsquigarrow \mathsf{K}\,A$. Now we use lemma 5.15 and need to prove $\vdash \mathsf{K}\,(l \rightsquigarrow \mathsf{K}\,A)$. Again we unshift and need to prove $l \vdash \mathsf{K}\,A$, which by co-reflection reduces to $l \vdash A$ which is our assumption. □

One result, we will later need when showing rules of the sequent calculus admissible is the following rule:

Lemma 5.17 *If* $\Gamma, \Delta, \mathsf{K}\,\Delta \vdash \mathsf{K}\,A$ *then* $\Gamma, \mathsf{K}\,(\Delta) \vdash A$.

Proof. We first duplicate the $\mathsf{K}\,\Delta$ part using weakening and then shift one instance of $\mathsf{K}\,\Delta$ to the right. Using Lemma 5.15 we now need to show $\Gamma, \mathsf{K}\,\Delta \vdash K(\Delta \rightsquigarrow A)$. We can use the co-reflection rule and un-shift and thus need to prove $\Gamma, \Delta, \mathsf{K}\,\Delta \vdash A$, which is our assumption. □

5.3 *Kripke completeness*

We will now showcase the modifications required to the canonical model construction. We start by introducing the modified kripke semantics.

Definition 5.18 (Modified semantics) *Let $\mathcal{M} = (\mathcal{W}, \mathcal{V}, \leq, \leq_{\mathbf{K}})$ be an* IEL *or* IEL$^-$*-model. We define the modified entailment relation $\Vdash'$ by induction on the formula.*

$$\begin{aligned}
\mathcal{M}, w \Vdash' p_i &:\Leftrightarrow \neg\neg\mathcal{V}(w, i)\\
\mathcal{M}, w \Vdash' A \wedge B &:\Leftrightarrow \mathcal{M}, w \Vdash' A \wedge \mathcal{M}, w \Vdash' B\\
\mathcal{M}, w \Vdash' A \supset B &:\Leftrightarrow \forall w'. w \leq w' \to \mathcal{M}, w \Vdash' A \to \mathcal{M}, w \Vdash B\\
\mathcal{M}, w \Vdash' A \vee B &:\Leftrightarrow \neg\neg\left(\mathcal{M}, w \Vdash' A \vee \mathcal{M}, w \Vdash' B\right)\\
\mathcal{M}, w \Vdash' \mathsf{K}\, A &:\Leftrightarrow \forall w', w \leq_{\mathbf{K}} w' \to \mathcal{M}, w \Vdash' A
\end{aligned}$$

When comparing the modified semantics with the unmodified one, the only differences are in the cases for disjunction and propositional variables.

The main modification we make is, that the worlds in the canonical models will be quasi-prime consistent deductively-closed theories.

Lemma 5.19 (Monotonicity) *The modified forcing relation $\Vdash'$ is monotone w.r.t $\leq$.*

Proof. We consider the propositional variable and disjunction case.

Case $A_1 \vee A_2$: We have $w_1 \Vdash' A_1 \to w_2 \Vdash' A_2$ and $w_1 \Vdash' A_2 \to w_2 \Vdash A_2$ as inductive hypotheses. We know $w_1 \Vdash' A_1 \vee A_2$ and need to show $w_2 \Vdash' A_1 \vee A_2$ which is the same as $\neg\neg(w_2 \vDash' A_1 \vee w_2 \vDash' A_2)$. Since the goal is negated we can reason classically and thus strip-off the double negations of $\neg\neg(w_1 \Vdash' A_1 \vee w_1 \Vdash' A_2)$. This leaves us with $\neg(w_2 \Vdash' A_1 \vee w_2 \Vdash' A_2)$ as an additional assumption and we still need to derive a contradiction. Using the inductive hypotheses we can obtain a proof of $w_2 \Vdash' A_1 \vee w_2 \Vdash' A_2$ which suffices to complete the proof.

Case p_i: We have $\neg\neg\mathcal{V}(w_1, p_i)$ and $\neg\mathcal{V}(w_2, p_i)$ as assumptions and need to derive a contradiction. Reasoning classically we can eliminate the double-negation. Using the monotonicity of Kripke-models w.r.t the valuation, we obtain $\mathcal{V}(w_2, p_i)$ which contradicts our assumption. □

We will now present the proof of the modified truth lemma.

Lemma 5.20 (Truth lemma) *For all worlds (which are theories) $\mathcal{T}$ in the canonical model, $\mathcal{T} \Vdash' A$ if and only if $\neg\neg A \in \mathcal{T}$.*

Proof. As in the classical case the proof is by induction on A.

Case p_i: Since $\mathcal{T} \Vdash' p_i \coloneqq \neg\neg\mathcal{V}(\mathcal{T}, p_i)$, which in the canonical model is equivalent to $\neg\neg p_i \in \mathcal{T}$, the proof is elementary (i.e. the terms are computationally equal).

Case $A_1 \vee A_2$: For the if-direction assume $\mathcal{T} \Vdash' A_1 \vee A_2$. Let us first assume $\mathcal{T} \Vdash' A_0 \vee A_1$, thus we have a proof of $\neg\neg(\mathcal{T} \Vdash' A_0 \vee \mathcal{T} \Vdash' A_1)$ and need to show $\neg\neg(A_0 \vee A_1 \in \mathcal{T})$. So we can assume $A_0 \vee A_1 \notin \mathcal{T}$ and need to produce a proof of contradiction. Since we have to prove falsity, we can strip the double negation in the assumption and thus obtain a proof $\mathcal{T} \Vdash' A_0 \vee \mathcal{T} \Vdash' A_1$. We do a case distinction on this.

- If $\mathcal{T} \Vdash' A_0$, we can conclude $\neg A_0 \notin \mathcal{T}$ by the inductive hypothesis. Since we still need to derive a contradiction, we can strip this double negation and have $A_0 \in \mathcal{T}$ as an assumption. To finally derive the contraction we can now show $A_0 \vee A_1 \in \mathcal{T}$. Since $A_0 \in \mathcal{T}$, we know that $\mathcal{T} \vdash A_0$ (this is the theory property of Lindenbaum-extensions). But then it is easy to show $\mathcal{T} \vdash A_0 \vee A_1$, using the left introduction rule for disjunction. Again using the theory property we have a proof of $A_0 \vee A_1 \in \mathcal{T}$ contradicting our assumption.
- This case is similar to the one before, the only difference is that we use the right-rule for disjunction.

For the other direction, assume $\neg\neg(A_1 \vee A_2 \in \mathcal{T})$ and we need to prove $\neg\neg(\mathcal{T} \Vdash' A_1 \vee \mathcal{T} \Vdash' A_2)$. Thus we can assume $\neg(\mathcal{T} \Vdash' A_1 \vee \mathcal{T} \Vdash' A_2)$ and need to derive a contradiction. Since we need to prove a contraction, we can strip the double negation in our assumption and thus can assume $A_1 \vee A_2 \in \mathcal{T}$. We can also strip the double-negation off the quasi-primeness property for $A_1 \vee A_2$, thus we know that $\mathcal{T} \vdash A_1 \vee A_2 \supset \mathcal{T} \vdash A_1 \vee \mathcal{T} \vdash A_2$. Thus we either have $\mathcal{T} \vdash A_1$ or $\mathcal{T} \vdash A_2$. In both cases, we can derive the contradiction from our assumption that $\neg(\mathcal{T} \Vdash' A_1 \vee \mathcal{T} \Vdash' A_2)$ and the inductive hypothesis. □

Instead of proving strong completeness, we can now prove strong quasi-completeness constructively.

Lemma 5.21 (Strong quasi-completeness) *If $\mathcal{T} \Vdash A$ then $\neg\neg\mathcal{T} \vdash A$.*

With decidability we will later obtain completeness. While we were able to prove strong quasi-completeness constructively we can only soundness using the law of excluded middle.

Lemma 5.22 (Soundness using LEM) *If $\mathcal{T} \vdash A$ then $\mathcal{T} \Vdash' A$.*

Proof. The proof is analogous to the proof for the usual semantics. However we need to use the law of excluded middle for the case of the disjunction elimination rule. □

One important thing to note is that we really need the full law of excluded middle here; we cannot show prime-soundness for the modified semantics.

6
Basic proof theory of IEL

The main result we will prove in this chapter is the decidability of IEL (and IEL^-). The fact that IEL is decidable is not a new result. It has already been established by Wolter and Zakharyaschev (1999), who prove, among other results, a class of intuitionistic modal logics decidable, one of which happens to be IEL.[1] Furthermore, PSPACE-completeness and thus also decidability was established for IEL (and IEL^-) by Krupski and Yatmanov (2016). In this chapter, we develop a mechanized decidability procedure for IEL basing on this work, however we will only obtain decidability but not PSPACE-completeness as a result.[2]

[1] Of course, IEL with its philosophical motivation was not considered there. They obtain these results using filtration methods, which are usually nonconstructive.

[2] The complexity of our algorithm is exponential, see the further work section for more on this.

We will not be able to do a proof search in the natural deduction system. Informally speaking, this is due to having too many degrees of freedom when choosing which rule to apply next. Especially interesting is the case of the implication elimination rule (or modus ponens):

$$\frac{\Gamma \vdash B \supset A \qquad B, \Gamma \vdash A}{\Gamma \vdash A}$$

The main problem with this rule is that it is in some sense universally applicable: In theory it can be applied anywhere in any derivation and the choice of formula B is completely arbitrary. If we want to do some kind of backwards proof search, this is bad, since we might need to try every possible value of B. Thus, instead of doing a proof search in the natural deduction system, we will use a sequent calculus without the cut-rule (the cut-rule roughly corresponds to implication elimination or modus ponens). Additionally, the sequent calculus representation has the subformula property. This means that any derivation of $\Gamma \Rightarrow A$ will only consist of subformulas of formulas in Γ, A. Fulfilling this property will guarantee the termination of the proof search. The natural deduction implication elimination rule does not fulfill the subformula property, since the formula B need not be a subformula of Γ, A. Thus our decidability proof has two steps: We first need to show that both systems agree, in the sense that $\Gamma \Rightarrow A$ if and only if $\Gamma \vdash A$. The second step is then to design a terminating proof search for the sequent calculus.

While we do not need to include the cut-rule in the sequent calculus to simulate the natural deduction rules, including modus ponens, we will still need to show it admissible:

$$\frac{\Gamma \Rightarrow B \qquad B, \Gamma \Rightarrow A}{\Gamma \Rightarrow A}\text{CUT}$$

If we had this rule in our sequent calculus , and would not just show it admissible, the sequent calculus would no longer have the subformula property. Sequent calculi were invented by Gentzen (1935a, 1935b) in his study of natural deduction.

Methodology For IEL (and IEL$^-$), cut-elimination was proven in Krupski and Yatmanov (2016). Their sequent calculus is an extension of the well-known sequent calculus G3I from the standard textbook by Troelstra and Schwichtenberg (2000). Krupski and Yatmanov extend the G3I calculus with two rules, one covering the K-introduction and distribution rules, and another rule equivalent to the intuitionistic reflection rule. Omitting the last rule, one obtains a sequent calculus for IEL$^-$.

We use these new rules and combine them with a sequent calculus introduced by Smolka and Brown (2012) and later extended to all standard connectives by Dang (2015) to obtain a cut-free sequent calculus which is well-suited for mechanization in the Coq proof assistant. This calculus is a variaton of the GKi-calculus from Troelstra and Schwichtenberg (2000). [3] The main difference between the calculi by Dang; Smolka and Brown and the GKI calculus is that the former uses membership prominently. Before coming to this final formulation, we also considered directly using the calculus from Krupski and Yatmanov (2016) with a permutation based embedding. While we were able to prove the results this way, there was a heavy mechanization overhead and less automation could be used. We will compare some approaches in Section 6.6. Throughout this chapter we will often do proofs by inductions on derivations, we will only prove selected cases in detail here, however all cases have been mechanized and checked in Coq. As in the preceeding chapters theorems are linked to their respective Coq formulization.

[3] Troelstra and Schwichtenberg attribute this calculus to Kleene (1952), thus the name GKi.

The layout of this chapter is as follows: We first introduce the sequent calculus we will use and prove some structural results (*i.e.* weakening and admissibility of inversion rules). Next, we detail the proof of cut-elimination, which relies on the results proven before. We can then prove the equivalence of the sequent calculus and the natural deduction system. Last, we prove decidability of the sequent calculus using a finite closure iteration. Thus we obtain the decidability of IEL and IEL$^-$. Afterwards, we take a look at a case-study extending the cut-elimination and decidability proofs to a classical modal logic, namely logic K. We close this chapter by comparing our approach with other possible approaches.

6.1 *Sequent representation for IEL*

The sequent calculus for IEL and IEL$^-$ is represented as an inductive predicate $\Rightarrow : \mathcal{L}(\mathcal{F}) \to \mathcal{F} \to \mathbb{P}$, the rules are given in Figure 6.1.1. What makes this calculus well-suited for mechanization is its heavy use of membership, which is easier to reason about in a proof assistant. Another distinctive property is that all rules are cumulative, since the antecedents of all premises only extend the context of the derived sequents.

Figure 6.1.1: Sequent system for IEL (IELD)

$$\frac{p_i \in \Gamma}{\Gamma \Rightarrow p_i}\ (V) \qquad \frac{\bot \in \Gamma}{\Gamma \Rightarrow S}\ (F)$$

$$\frac{F, \Gamma \Rightarrow G}{\Gamma \Rightarrow F \supset G}\ (IR) \qquad \frac{F \supset G \in \Gamma \qquad \Gamma \Rightarrow F}{\Gamma \Rightarrow G}\ (IL)$$

$$\frac{F \wedge G \in \Gamma \qquad F, G, \Gamma \Rightarrow H}{\Gamma \Rightarrow H}\ (AR) \qquad \frac{\Gamma \Rightarrow F \qquad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \wedge G}\ (AL)$$

$$\frac{F \vee G \in \Gamma \qquad F, \Gamma \Rightarrow H \qquad G, \Gamma \Rightarrow H}{\Gamma \Rightarrow H}\ (OL) \qquad \frac{\Gamma \Rightarrow F_i}{\Gamma \Rightarrow F_1 \vee F_2}\ (\mathrm{AR}_i)$$

$$\frac{\Gamma, \mathsf{K}^{-}(\Gamma) \Rightarrow F}{\Gamma \Rightarrow \mathsf{K}F}\ (\mathrm{KI}) \qquad \frac{\Gamma \Rightarrow \mathsf{K}\bot}{\Gamma \Rightarrow A}\ (\mathrm{KF})$$

By $\Gamma \overset{n}{\Rightarrow} A$ we denote, that A can be derived from the multiset Γ using n steps or less; we encode this as an inductive predicate too. Our height-encoding is the same as the one used by Michaelis and Nipkow (2017); that is, we always assume that the subderivations' heights are equal. Once we include an additional rule to increase any derivations height by one, the resulting system is equivalent to the one obtained by taking the maximum height over the subderivations' heights increased by one as the height of a derivation. Let us consider the left introduction rule for conjunction. Using a minimum maximum encoding for height, one would have the following rule in the calculus:

$$\frac{\Gamma \overset{h_1}{\Rightarrow} F \qquad \Gamma \overset{h_2}{\Rightarrow} G}{\Gamma \overset{\max(h_1,h_2)+1}{\Rightarrow} F \wedge G}$$

We encode it in the following way:

$$\frac{\Gamma \overset{h}{\Rightarrow} F \qquad \Gamma \overset{h}{\Rightarrow} G}{\Gamma \overset{h+1}{\Rightarrow} F \wedge G}$$

We can now prove an obvious correspondence result between the system with and without heights.

Lemma 6.1 (Characterization of the height system) *$\Gamma \Rightarrow A$ if and only if there exists an n such that $\Gamma \overset{n}{\Rightarrow} A$.*

Proof. Both cases are proven by induction on the derivation. □

Note that strictly speaking, the intuitionistic reflection rule (KF) in the sequent calculus does not fulfill the subformula property, since $\mathsf{K}\bot$ need not be a subformula of A, Γ.[4] However, for our proof search it will suffice to bound the set of possible antecedents in a derivation which is still possible.

[4] Su and Sano (2019) present a sequent calculus for IEL with the subformula property. This is achieved by allowing empty sets of formulas in the succedent, thus it would be harder to mechanize.

[5] Since the calculus for IEL excludes the KF-rule, this statement could be stronger: It enjoys the subformula property.

Lemma 6.2 *The sequent calculus for IEL (and* IEL$^-$*)*[5] *has the (pseudo)-subformula property. That is, in any derivation $\Gamma \Rightarrow A$ only subformulas of formulas contained in $\mathsf{K}\bot, A, \Gamma$ occur.*

Let us now start proving some technical results. We will start by proving depth-preserving weakening.

Lemma 6.3 (Weakening) *If $\Gamma \subseteq \Omega$ and $\Gamma \overset{n}{\Rightarrow} A$ then $\Omega \overset{n}{\Rightarrow} A$.*

Proof. Many cases in this proof are elementary, this is due to the use of only membership and subderivations in the formulation. Since $A \in \Gamma$ and $\Gamma \subseteq \Omega$, all of those cases are pretty easy. We will take a look at the variable (V) , left rule for conjunction (AL) and K-introduction (KI) cases now.

Case V: Assume $p_i \in \Gamma$ and $\Gamma \subseteq \Omega$. We need to prove $\Omega \overset{0}{\Rightarrow} p_i$. We can apply the V rule and now only need to show $p_i \in \Omega$, which we can, since $\Gamma \subseteq \Omega$.

Case AL: Assume $\Gamma \overset{n}{\Rightarrow} A$ was derived using the AL-rule, thus $B_1 \wedge B_2 \in \Gamma$ and we have a derivation $B_1, B_2, \Gamma \overset{n-1}{\Rightarrow} A$. We construct the following derivation:

$$\dfrac{B_1 \wedge B_2 \in \Omega \qquad \dfrac{B_1, B_2, \Gamma \overset{n-1}{\Rightarrow} A}{B_1, B_2, \Omega \overset{n-1}{\Rightarrow} A}\text{weak. (I.H.)}}{\Omega \overset{n}{\Rightarrow} A}\text{AL}$$

Case KI: We essentially use, that $\Gamma, \mathsf{K}^-(\Gamma) \subseteq \Omega, \mathsf{K}^-(\Omega)$ since $\Gamma \subseteq \Omega$. Then the case is solved, just like the other cases. □

Readers familiar with the structural cut-elimination proof in textbooks like Troelstra and Schwichtenberg (2000) might expect us to next prove a contraction result. However, our weakening property is stronger than textbook weakening and already absorbs the contraction property. Most textbooks prove

$$\Gamma \Rightarrow A \to B, \Gamma \Rightarrow A$$

as the weakening property, but our property is stronger. To see this, consider $\{A, A, A\}$ and $\{A\}$. Using textbook weakening one can not obtain a proof of $\{A\} \overset{n}{\Rightarrow} B$ from $\{A, A, A\} \overset{n}{\Rightarrow} B$, while this is possible using our weakening result. The textbook weakening does not allow to contract double occurrences of formulas, while our version using inclusion allows for this. Proving this stronger property directly is only possible due to the membership based representation.

We additionally need to prove inversion results, allowing us to invert some rules of the calculus.

Lemma 6.4 (Inversion results) *The rules for conjunction, disjunction and implication are height-preservingly invertible in the following sense:*

- $A \in \Gamma \to A \supset B, \Gamma \overset{n}{\Rightarrow} C \to \Gamma \overset{n}{\Rightarrow} C$
- $A \in \Gamma \to A \vee B, \Gamma \Rightarrow C \to \Gamma \overset{n}{\Rightarrow} C$
- $B \in \Gamma \to A \vee B, \Gamma \Rightarrow C \to \Gamma \overset{n}{\Rightarrow} C$
- $A \in \Gamma \to B \in \Gamma \to A \wedge B, \Gamma \overset{n}{\Rightarrow} C \to \Gamma \Rightarrow C$

Proof. The proofs are by induction on the height with the formulas quantified (just as in Troelstra and Schwichtenberg (2000))[6]. Most cases are solved by applying the rule used to obtain the derivation and using the inductive hypothesis afterwards. Only when the rule we are showing invertible is used on the same formulas (e.g. same A and B), it suffices to use the inductive hypothesis directly. □

[6] Indrzejczak (2021) attribute this method (e.g. induction on the height) of proving inversion results to Schütte (1977). Interestingly, Michaelis and Nipkow (2017) prove these results by induction on the derivation in Isabelle/Hol, however in Coq we had no success with inductions on the derivation here.

Before coming to the main proof of this section, the cut-elimination proof, we can prove the disjunction property for the sequent-calculus.

Lemma 6.5 (SC Disjunction property) *If* $\Rightarrow A \vee B$ *then* $\Rightarrow A$ *or* $\Rightarrow B$*.*

Proof. The proof is by induction on the derivation $\Rightarrow A \vee B$. □

6.2 *Cut-Elimination proof*

The general idea of the cut-elimination proof we present here follows textbooks proofs (e.g. Troelstra and Schwichtenberg (2000), Indrzejczak (2021)).[7] However, textbooks can make distinctions based on principality, roughly speaking, wether two derivations rules match in the sense that they use the corresponding introduction and elimination rule of a logical connective or not. Since many non-principal cases are similar these each need not be discussed individually. In a proof-assistant we cannot use such reasoning, and instead need to rely more heavily on inversions *i.e.* case analyses on the derivations.

[7] Indrzejczak (2021) attributes this cut-elimination proof for G3K to Dragalin (1987).

The general strategy is, to define a complexity measure on cuts and subsequently eliminate cuts by replacing them with cuts of a lower complexity, which can be eliminated using the inductive hypothesis. We start by showing the induction principle we will use to prove the result:

Lemma 6.6 (Lexicographic Pair Induction) *Let* $P : \mathbb{N} \to \mathbb{N} \to \mathbb{P}$ *be a predicate. The following induction principle is valid:*

$$(\forall n\, m.\, (\forall p\, q.\ p < n \to P\, p\, q) \to (\forall q. q < m \to P\, n\, q) \to P\, n\, m) \to \forall nm. P\, n\, m$$

Proof. Using two nested strong inductions, first on n, afterwards on m. □

Theorem 6.7 (Cut is admissible for IEL and IEL⁻) *The cut rule is admissible for both* IEL *and* IEL$^-$.

$$\dfrac{\overset{[\delta_1]}{\Gamma \overset{h_1}{\Rightarrow} B} \qquad \overset{[\delta_2]}{B, \Gamma \overset{h_2}{\Rightarrow} A}}{\Gamma \Rightarrow A}\; Cut$$

Proof. The version of cut we prove here is the so-called *context-sharing* version, since the multiset Γ occurs in both derivations. There would be the possibility to instead use two distinct contexts for the left and right side, however this makes the proofs more complicated. Note, that in calculi with (strong) weakening, both are equivalent, we will see this in Section 6.6. The proof is by induction on pairs (s, r) of formula-size s and cut-rank r. Here formula size is the size (c.f. Definition 4.2) of the cut-formula B, and the cut-rank is the sum of the heights i.e. $r := h_1 + h_2$.

As an induction principle, we use the pair induction from Lemma 6.6. This induction principle gives us two inductive hypotheses, one which allows us to eliminate cuts of arbitrary height but with a cut formula of smaller size (s-cut) and another one, allowing us to eliminate cuts on formulas of the same size but with a smaller cut-rank (r-cut).

We now analyze which rule was used to derive δ_1. In two cases, namely the K introduction and right implication introduction rule we will need an additional case analysis (i.e. inversion) on δ_2.

AL-Rule: Assume δ_1 was derived using the left-rule for conjunction. Our derivation has the following form.

$$\dfrac{\dfrac{C_1 \wedge C_2 \in \Gamma \qquad C_1, C_2, \Gamma \overset{n-1}{\Rightarrow} F}{\Gamma \overset{n}{\Rightarrow} B} \qquad B, \Gamma \overset{m}{\Rightarrow} A}{\Gamma \Rightarrow A}\; \text{r-cut}$$

We can permute the application of the left rule for conjunction downwards and use weakening on the derivation $C_1, C_2, \Gamma \overset{m}{\Rightarrow} \Delta$:

$$\dfrac{C_1 \wedge C_2 \in \Gamma \qquad \dfrac{C_1, C_2, \Gamma \overset{n-1}{\Rightarrow} B \qquad B, \Gamma \overset{m}{\Rightarrow} A}{C_1, C_2, \Gamma \Rightarrow A}}{\Gamma \Rightarrow A}$$

Note that the new cut is a cut on the same formula but of a smaller rank, thus we can eliminate it by the inductive hypothesis.

IR-Rule: Assume last rule used in the derivation of δ_1 was the right introduction rule for implication. Thus we know, that $B = B_1 \supset B_2$. We need to do a second case analysis on the derivation δ_2.

1. If δ_2 is an axiom, either $p_i = B$ or $p_i \in \Gamma$ and we know that $A = p_i$. The first case contradicts our assumption that $B = B_1 \supset B_2$ and in the second case we can directly use the variable rule.
2. Similarly, if the second premiss is derived using the falsity rule, either $F = \bot$ or $\bot \in \Gamma$.

3. An interesting case arises when the right premiss is proved using the left introduction rule for implication.

$$\dfrac{\dfrac{B_0,\Gamma \overset{h_1-1}{\Rightarrow} B_1}{\Gamma \overset{h_1}{\Rightarrow} B_0 \supset B_1} \qquad \dfrac{C_0\supset C_1 \in B,\Gamma \quad B,\Gamma \overset{h_2-1}{\Rightarrow} C_0 \quad C_1,B,\Gamma\Rightarrow A}{B_0\supset B_1 \overset{h_2}{\Rightarrow} B}}{\Gamma \Rightarrow A}$$

We have 2 cases: Either $B = C_0 \supset C_1$ or $C_0 \supset C_1 \in \Gamma$.

(a) In the first case, we can build the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma \overset{h_1}{\Rightarrow} B \quad B,\Gamma \overset{h_2-1}{\Rightarrow} B_0}{\Gamma\Rightarrow B_0}\text{r-cut} \quad B_0,\Gamma\Rightarrow B_1}{\Gamma\Rightarrow B_1} \qquad \dfrac{B_1\in B_1,\Gamma \quad B_1,B,\Gamma\Rightarrow A}{B_1,\Gamma\Rightarrow A}\text{IL-inv}}{\Gamma\Rightarrow A}$$

(b) In the second case, we can apply the left rule for implication first and do two cuts afterwards.

$$\dfrac{C_0\supset C_1\in\Gamma \qquad \dfrac{\Gamma \overset{h_1}{\Rightarrow} B \quad B,\Gamma \overset{h_2-1}{\Rightarrow} C_0}{\Gamma\Rightarrow C_0}\text{r-cut} \qquad \dfrac{\dfrac{\Gamma \overset{h_1}{\Rightarrow} B}{C_1,\Gamma \overset{h_1}{\Rightarrow} B}\text{weak.} \quad B,C_1,\Gamma \overset{h_2-1}{\Rightarrow} C_0}{C_1,\Gamma\Rightarrow A}\text{r-cut}}{\Gamma\Rightarrow A}\text{IL}$$

KI-Rule: Assume the premiss was derived using the K-introduction rule. We need to make a second case distinction on the derivation of the right deduction. Most cases are similar to those obtained in the right rule for implication subcases, and we will not go into too much detail here.

$$\dfrac{\dfrac{\Gamma,\mathsf{K}^-(\Gamma)\Rightarrow B_0}{\Gamma\Rightarrow \mathsf{K}\,B_0} \quad \mathsf{K}\,B_0,\Gamma\Rightarrow A}{\Gamma\Rightarrow A}$$

1. The right premise is an axiom. Either $p_i = \mathsf{K}\,B_0$ which is impossible (since the constructors of an inductive datatype are disjoint) or $A \in \Gamma$ in which case we can directly construct the derivation.
2. The most interesting case occurs when the KI-rule is used on both sides. We have the following derivation:

$$\dfrac{\dfrac{\Gamma,\mathsf{K}^-(\Gamma)\overset{h_1-1}{\Rightarrow}B_0}{\Gamma\overset{h_1}{\Rightarrow}\mathsf{K}\,B_0} \quad \dfrac{\mathsf{K}\,B_0,B_0,\Gamma,\mathsf{K}^-(\Gamma)\overset{h_2-1}{\Rightarrow}A_0}{\mathsf{K}\,B_0,\Gamma\overset{h_2}{\Rightarrow}\mathsf{K}\,A_0}}{\Gamma\Rightarrow\mathsf{K}\,A_0}$$

We can build the following derivation:

$$\dfrac{\dfrac{\Gamma,\mathsf{K}^-(\Gamma)\Rightarrow B_0 \quad \dfrac{\dfrac{\Gamma,\mathsf{K}^-(\Gamma)\overset{h_1}{\Rightarrow}\mathsf{K}\,B_0}{\Gamma,\mathsf{K}^-(\Gamma)\overset{h_1}{\Rightarrow}\mathsf{K}\,B_0}\text{weak.} \quad \mathsf{K}\,B_0,B_0,\Gamma,\mathsf{K}^-(\Gamma)\overset{h_2-1}{\Rightarrow}A_0}{B_0,\Gamma,\mathsf{K}^-(\Gamma)\Rightarrow A_0}\text{r-cut}}{\Gamma,\mathsf{K}^-(\Gamma)\Rightarrow A_0}\text{s-cut}}{\Gamma\Rightarrow\mathsf{K}\,A_0}\text{KI}$$

6.3 *Equivalence between ND and SC*

As the section title suggests, we will now be concerned with proving the equivalence between the sequent calculus and the natural deduction representations. The general structure of these proofs is well known. While proving the direction transforming a sequent calculus derivation into a natural deduction one is fairly easy (the K-case is slightly more involved), the other direction is harder to proof and needs admissibility of cut.

Lemma 6.8 (SC to ND) *If* $\Gamma \Rightarrow A$ *then* $\Gamma \vdash A$

Proof. The proof is by induction on the derivation $\Gamma \Rightarrow A$, in Coq most cases are solved automatically. Easiest are the cases of right rules, which are basically the same in the sequent calculus and natural deduction. The left rules are also fairly easy, but use modus ponens in the natural deduction system quite often.

Case AL: We have the following sequent calculus derivation:

$$\frac{B_1 \wedge B_2 \in \Gamma \qquad B_1, B_2, \Gamma \Rightarrow A}{\Gamma \Rightarrow A}$$

As the inductive we obtain a derivation $B_1, B_2, \Gamma \vdash A$. We can build the following natural deduction derivation[8]:

$$\dfrac{\dfrac{\dfrac{B_1 \wedge B_2 \in \Gamma}{\Gamma \vdash B_1 \wedge B_2}}{\Gamma \vdash B_2} \qquad \dfrac{\dfrac{\dfrac{B_1 \wedge B_2 \in B_1, \Gamma}{B_1, \Gamma \vdash B_1 \wedge B_2}}{B_1, \Gamma \vdash B_2} \qquad B_1, B_2, \Gamma \vdash A}{B_2, \Gamma \vdash A}}{\Gamma \vdash A}$$

[8] We used the cut-rule in ND directly, instead of first using the elimination rule for implication followed by using the introduction rule.

Case OR_1: We have a the following sequent calculus derivation:

$$\frac{\Gamma \Rightarrow A \vee B}{\Gamma \Rightarrow A}$$

We can easily turn this into the following natural deduction derivation:

$$\frac{\Gamma \vdash A \vee B}{\Gamma \vdash A}$$

Case KI: We have the following sequent calculus derivation:

$$\frac{\Gamma, \mathsf{K}^{-}(\Gamma) \Rightarrow A_0}{\Gamma \Rightarrow \mathsf{K}\, A_0}$$

As was shown in Lemma 5.17, the following rule is admissible in IEL:

$$\Gamma, \Omega, \mathsf{K}\,\Omega \vdash A \to \Gamma, \mathsf{K}\,\Omega \vdash \mathsf{K}\, A$$

With this rule, the derivation is easy, we use that $\mathsf{K}\,(\mathsf{K}^{-}(\Gamma)) \subseteq \Gamma$:

$$\dfrac{\dfrac{\dfrac{\Gamma, \mathsf{K}^{-}(\Gamma) \Rightarrow A_0}{\Gamma, \mathsf{K}\,(\mathsf{K}^{-}(\Gamma), \mathsf{K}\,\Gamma}\,\text{(weak)}}{\Gamma, \mathsf{K}\,(\mathsf{K}^{-}(\Gamma)) \vdash \mathsf{K}\,A_0}}{\Gamma \vdash \mathsf{K}\,A_0}$$

For the other direction, we can first prove admissibility results.

Lemma 6.9 (ND-rules are admissible) *The following rules are admissible:*

- $\Gamma \Rightarrow \bot \to \forall A.\, A \in \Gamma$
- $A \in \Gamma \to \Gamma \Rightarrow A$
- $\Gamma \Rightarrow A \wedge B \to (\Gamma \Rightarrow A) \wedge (\Gamma \Rightarrow B)$
- $\Gamma \Rightarrow A \supset B \to A, \Gamma \Rightarrow B$

Proof. All of the proofs are by induction on the derivations.

Lemma 6.10 (From ND to SC) *If* $\Gamma \vdash A$ *then* $\Gamma \Rightarrow A$.

Proof. We use cut and the admissible rules multiple times. The variable and falsity cases can be solved rather easily using the admissibility results.

Case IE: As observed earlier, the implication elimination rule corresponds to cut.

$$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash A \supset B}{\Gamma \vdash B}$$

We build the following sequent calculus derivation:

$$\dfrac{\Gamma \Rightarrow A \qquad \dfrac{\Gamma \Rightarrow A \supset B}{A, \Gamma \Rightarrow B}\,\text{admissibility}}{\Gamma \Rightarrow B}\,\text{cut}$$

As a corollary, we obtain the equivalence result.

Corollary 6.11 $\Gamma \vdash A$ *if and only if* $\Gamma \Rightarrow A$.

Proof. Immediate with Lemma 6.10 and Lemma 6.8. □

As a corollary we now have a constructive proof of the disjunction property for IEL and IEL^{-}.

Lemma 6.12 (Disjunction Property) *If* $\vdash A \vee B$ *then either* $\vdash A$ *or* $\vdash B$.

Proof. We use the equivalenece from Corollary 6.11 to replace natural deduction entailment with sequent calculus entailment. The proof is finished using the disjunction property for the sequent calculus (*i.e.* Lemma 6.5). □

6.4 Decidability of IEL

We now prove decidability of IEL (and IEL^{-}) utilizing a proof-search procedure in the cut-free sequent calculus. Throughout this section we fix a finite multiset (i.e. list in the formalization) Γ and a formula A and want to a design decision procedure which decides wether $\Gamma \Rightarrow A$ or $\Gamma \not\Rightarrow A$. Since Γ and A are arbitrary, this will give us the decidability proof.

The general strategy for the decidability proof is to compute the set of derivable sequents which are only made-up of subformulas of Γ and A. For this let $\mathcal{U}$ denote the list of subformulas from Γ and A and $\mathsf{K}\,\bot$, which can be computed. The following set (i.e. a list in the formalization)

$$\left\{(\Gamma', A')\middle|\Gamma' \Rightarrow A' \wedge \Gamma \subseteq \mathcal{U} \wedge A' \in \mathcal{U}\right\}$$

, containing every derivable $\mathcal{U}$-sequent, can be computed using a fixed-point iteration, incrementally expanding the set of derivable sequents. This approach is possible, since we have a (modified) subformula property and the universe is finite.

We call a pair (Γ, A) a goal and more specifically a U-goal, if $\Gamma \subseteq U$ and $A \in U$. Let us call a pair (Γ, A) normalized, if the elements of Γ are ordered just like the elements in U. With $\ulcorner\Gamma\urcorner$ we denote the list obtained by reordering the elements from Γ just like the elements from U. In the proof search we can restrict the search to normalized U-goals due to weakening.

We now wish to construct a list of derivable U-goals. This list will be constructed using a fixed point iteration. In each step of the iteration, we will know a set (in the formalization a list) of goals, which can be derived and need to check if we can extend this by a specific sequent. For example, if we know that $\Gamma \Rightarrow A$ and $\Gamma \Rightarrow B$, because (Γ, A) and (Γ, B) are contained in the list of derived sequents, we know that $\Gamma \Rightarrow A \wedge B$ is derivable and thus (if $A \wedge B \in U$) we could add the pair $(\Gamma, A \wedge B)$ to the list.

Lemma 6.13 (c.f. Smolka2012, Lemma 12.3.3) *One can construct a list Λ of U-goals such that:*

1. *If $(\Gamma, A) \in \Lambda$ then $\Gamma \Rightarrow \Lambda$*
2. *Λ contains every U-goal (Γ, A) satisfying:*
   - $\bot \in \Gamma$
   - *$B \supset C \in \Gamma$ and $(\Gamma, B) \in \Lambda$ and $(\ulcorner C, \Gamma\urcorner, A) \in \Lambda$.*
   - *$B \wedge C \in \Gamma$ and $(\ulcorner B, C, \Gamma\urcorner, A) \in \Lambda$*
   - *$B \vee C \in \Gamma$ and both $(\ulcorner B, \Gamma\urcorner, A)$ and $(\ulcorner C, \Gamma\urcorner, A) \in \Lambda$.*
   - *$A = p_i$ and $p_i \in \Gamma$*
   - *$A = A_1 \supset A_2$ and $(\ulcorner\Gamma, A_1\urcorner, A_2) \in \Lambda$*
   - *$A = A_1 \wedge A_2$ and $(\Gamma, A_1) \in \Lambda$ and $(\Gamma, A_2) \in \Lambda$*

- $A = A_1 \vee A_2$ *and* $(\Gamma, A_1) \in \Lambda$ *or* $(\Gamma, A_2) \in \Lambda$
- $A = \mathsf{K}\, A_1$ *and* $(\ulcorner \mathsf{K}^-(\Gamma) \urcorner, A_1) \in \Lambda$

Proof. This list can be constructed using a fixed-point construction. □

What we need to prove decidability is to verify that using the rules in (2) as a step-function starting from the empty list, we will be able to compute all of the subformulas which can be computed.

Our presentation of this method follows Menz (2016) and Smolka and Brown (2012). In the general setting, we want to compute a list with elements from an enumerable universe V, with elements from a decidable type X that satisfy a property $p : X \to \mathbb{P}$. For this, we assume to have constructed a step predicate $\mathsf{step} : \mathcal{L}(X) \to X \to \mathbb{P}$, which is decidable. In each step, we check if there is an element u from the universe and check whether $s\, L\, u$ holds. If this is the case, we add u to the list, otherwise we do not modify the list. One can easily verify that this process yields a fixed point after at most $|V|$ iterations.[9] However, we still need to establish the connection between this fixed-point and the derivability for the sequent calculus. We define (following Menz) $A \subseteq p$ for a list A over X as $\forall x \in A.\, p\, x$.

[9] Since the step function is deterministic, if there have been no elements added during an iteration, there will never be in the next round. Thus after $|\mathcal{V}|$ iterations the fixed-point will have been reached.

Two crucial properties we need in the proofs are the closure property and induction principle.

Lemma 6.14 *The following hold for the list* Λ *we construct by finite-closure iteration:*

- Λ*-Closure. If* $\mathsf{step}\, \Lambda\, x$ *and* $x \in \mathcal{U}$*, then* $x \in C$.
- Λ*-Induction. Let* $\mathsf{step}\, A\, x \to p\, x$ *for all* $A \subseteq p$ *and* $x \in V$*. Then* $\Lambda \subseteq p$.

Proof. See Lemma 12.4.2 in Smolka and Brown (2012). □

Lemma 6.15 *If* $\Gamma \subseteq \mathcal{U}$ *and* $A \in \mathcal{U}$ *and* $\Gamma \Rightarrow A$ *then* $(\ulcorner \Gamma \urcorner, A) \in \Lambda$.

Proof. The proof is by induction on $\Gamma \Rightarrow A$. We use closure in every step and thus only need to prove that $\mathsf{step}\, \Lambda\, (\ulcorner \Gamma \urcorner, A)$ holds.

Case AR: Assume $\Gamma \Rightarrow A_1$ and $\Gamma \Rightarrow A_2$, thus $(\ulcorner \Gamma \urcorner, A_1) \in \Lambda$ and $((\ulcorner \Gamma \urcorner, A_2) \in \Lambda$ by the inductive hypothesis. Thus by the definition of step, $\mathsf{step}\, \Lambda\, (\ulcorner \Gamma \urcorner, A_1 \wedge A_2)$. Generally speaking, all the right rules are easily solved similarly.

Case AL: Assume there is $B \wedge C \in \Gamma$ and $B, C, \Gamma \Rightarrow A$. Thus by the inductive hypothesis $(\ulcorner B, C, \Gamma \urcorner, A) \in \Lambda$ (of course, we need to show that $B, C, \Gamma \subseteq \mathcal{U}$ but this simple since $\mathcal{U}$ is subformula-closed). By using the third rule from Lemma 6.13 we need to show $(\ulcorner B, C, \ulcorner \Gamma \urcorner \urcorner, A) \in \Lambda$, but since for $\ulcorner B, C, \ulcorner \Gamma \urcorner \urcorner = \ulcorner B, C, \Gamma \urcorner$ this is our inductive hypothesis. □

Lemma 6.16 *If* $(\Gamma, A) \in \Lambda$ *then* $A \Rightarrow A$.

Proof. By Λ-induction. We can then do a case analysis on why the step-function was fulfilled and thereby prove the result. □

Lemma 6.17 (SC is decidable) *For any context* Γ *and formula* A,

$$(\Gamma \Rightarrow A) + (\Gamma \not\Rightarrow A).$$

Proof. Since the type of U-goals is decidable, we compute Λ and check if $(\ulcorner\Gamma\urcorner, A) \in \Lambda$. In the positive case, we obtain a proof of $\Gamma \Rightarrow A$ by Lemma 6.16. In the negative case, we can prove $\Gamma \not\Rightarrow A$ by deriving a contradiction. Thus we assume $\Gamma \Rightarrow A$, now using our assumption, it suffices to proof $(\ulcorner\Gamma\urcorner, A) \in \Lambda$, what we can do with Lemma 6.15.

Corollary 6.18 (ND is decidable) *For any context* Γ *and formula* A,

$$(\Gamma \vdash A) + (\Gamma \nvdash A).$$

Proof. This result is obtained by combining the equivalence proof between the sequent-calculus and natural deduction systems (Corollary 6.11) with Lemma 6.17. □

As hinted at in the chapter on the constructive strong quasi-completeness we can now obtain a constructive proof of completeness.

Corollary 6.19 (Completeness) *If* $\Gamma \Vdash' A$ *then* $\Gamma \vdash A$.

Proof. Assume $\Gamma \Vdash' A$. By quasi-completeness $\neg\neg\Gamma \vdash A$. We can now use the decider from Corollary 6.18 to do a case distinction on $\Gamma \vdash A$ or $\Gamma \nvdash A$. If $\Gamma \vdash A$ we are done. In the other case, we can derive a contradiction since we have both $\Gamma \nvdash A$ and $\neg\neg\Gamma \vdash A$ as assumptions.□

6.5 *The Classical Modal Logic K*

Last, we will showcase how to adapt this method of proving cut-elimination and decidability to the classical modal logic K (Kripke, 1959).

We chose the classical modal logic K here as a benchmark since it is the simplest classical modal logic. Logic K only has two rules additional to the rules of propositional logic:

- Necessitation Rule: $\vdash A \rightarrow \vdash \Box A$
- Distribution Rule: $\Box(A \supset B) \rightarrow \Box A \supset \Box B$

Note that on a formal level, IEL^- and K are pretty similar: Both have only two additional rules, however, while IEL^- has co-reflection as an axiom, K only has the necessitation rule.

The methods used for proving this should generalize to other modal logics (*e.g.* S4 or extensions of K as KT)[10].

For this, we extend the sequent calculus for classical propositional logic used by Dang (2015) and Smolka and Brown (2012) by the classical K-rule from Hakli and Negri (2012), the rules for this sequent calculus are given in Figure 6.5.1. Note, that as most sequent calculi for classical logic, we have a multiset of formulas in the succedent, too.

The modal rule we use is slightly different from the one given by Hakli and Negri. They use the following rule:

[10] There is a possibility to extend the G3C-calculus to S4 (Troelstra & Schwichtenberg, 2000).

$$\frac{p_i \in \Gamma \qquad p_i \in \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{A}) \qquad \frac{\bot \in \Gamma}{\Gamma \Rightarrow \Delta}\ (\mathrm{F})$$

$$\frac{A \supset B \in \Gamma \qquad \Gamma \Rightarrow A, \Delta \qquad B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{IL}) \qquad \frac{A \supset B \in \Delta \qquad A, \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{IR})$$

$$\frac{A \wedge B \in \Gamma \qquad A, B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{AL}) \qquad \frac{A \wedge B \in \Delta \qquad \Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{AR})$$

$$\frac{A \vee B \in \Gamma \qquad A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{OL}) \qquad \frac{A \vee B \in \Delta \qquad \Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{OR})$$

$$\frac{\Box A \in \Omega \qquad \Box^{-}(\Gamma) \Rightarrow A}{\Gamma \Rightarrow \Omega}\ (\mathrm{K})$$

Figure 6.5.1: Sequent system G3K

$$\frac{\Gamma \Rightarrow A}{\Box(\Gamma), \Theta \Rightarrow \Delta, \mathrm{K}\,A}$$

Note that A is a single formula and not a set of formulas. We essentially made two changes: First, we use the same trick as in the IEL case, where our rule is such, that a maximal Γ is chosen.[11] Second, we express the multiset-constraint on the succedent using membership. Because both systems enjoy height-preserving weakening, this does not change the derivable sequents capabilities.

[11] Krupski and Yatmanov propose the following introduction rule for K:

$$\frac{\Gamma, \Delta, \mathrm{K}\,\Delta \Rightarrow A}{\Gamma, \mathrm{K}\,(\Delta) \Rightarrow \mathrm{K}\,A}$$

The rule we chose (also mentioned by Krupski and Yatmanov) is the result of always choosing a maximal Δ.

Lemma 6.20 (Weakening for K) *If $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$ and it is possible to derive $\Gamma \Rightarrow_K \Delta$ then it is possible to derive $\Gamma' \Rightarrow_K \Delta'$.*

Proof. Just as in the intuitionistic case, by induction on the derivation of $\Gamma \Rightarrow \Delta$ with Γ', Δ' quantified. □

Following the cut-elimination proof for the intuitionistic case, we now state and prove inversion results.

Lemma 6.21 (Inversion principles for K) *The following inversion principles hold:*

- $A \in \Delta \to A \supset B, \Gamma \Rightarrow_K \Delta \to \Gamma \Rightarrow_K \Delta$
- $B \in \Gamma \to A \supset B, \Gamma \Rightarrow_K \Delta \to \Gamma \Rightarrow_K \Delta$
- $A \in \Gamma \to B \in \Gamma \to A \wedge B, \Gamma \Rightarrow_K \Delta \to \Gamma \Rightarrow \Delta$
- $A \in \Gamma \to B \in \Gamma \to \Gamma \Rightarrow_K A \vee B, \Delta \to \Gamma \Rightarrow \Delta$
- $A \in \Gamma \to A \vee B, \Gamma \Rightarrow_K \Delta \to \Gamma \Rightarrow \Delta$
- $B \in \Gamma \to A \vee B, \Gamma \Rightarrow_K \Delta \to \Gamma \Rightarrow \Delta$

Proof. All of the proofs are by induction on n and inversion on the derivation afterwards. □

Now, we can proceed to show the cut-elimination theorem for K.

Lemma 6.22 (Cut-Elimination for K) *The following rule is admissible in G3K.*

$$\frac{\Gamma \overset{h_1}{\Rightarrow} A,\Delta \qquad A,\Gamma \overset{h_2}{\Rightarrow} \Delta}{\Gamma \Rightarrow \Delta}$$

Proof. Just as in the intuitionistic case, we use the same pair-induction principle on the formula-size and cut-rank. We do an inversion on the derivation $\Gamma \Rightarrow A,\Delta$.

AL-Rule: Since the left rule for conjunction was used, the derivation has the following form:

$$\frac{\dfrac{B \wedge C \in \Gamma \qquad B,C,\Gamma \overset{h_1-1}{\Rightarrow} A,\Delta}{\Gamma \overset{h_1}{\Rightarrow} A,\Delta} \qquad A,\Gamma \overset{h_2}{\Rightarrow} \Delta}{\Gamma \Rightarrow \Delta}$$

Just as in the intuitionistic case, we can permute the cut upwards and first apply the AL-Rule and obtain the following derivation:

$$\frac{B \wedge C \in \Gamma \qquad \dfrac{B,C,\Gamma \overset{h_1-1}{\Rightarrow} A,\Delta \qquad A,\Gamma \overset{h_2}{\Rightarrow} \Delta}{B,C,\Gamma \Rightarrow A,\Delta}}{\Gamma \Rightarrow \Delta}$$

IR-Rule: This case is quite different from its intuitionistic counterpart, since we have stronger inversion results for the classical system, which spare us the second inversion in this case. The derivation has the following form:

$$\frac{\dfrac{B \supset C \in A,\Delta \qquad B,\Gamma \overset{h_1-1}{\Rightarrow}_D A,\Delta}{\Gamma \overset{h_1}{\Rightarrow} A,\Delta} \qquad A,\Gamma \overset{h_2}{\Rightarrow} \Delta}{\Gamma \Rightarrow \Delta}$$

We either have $A = B \supset C$ or $B \supset C \in \Delta$.

1. We can use the inversion lemmata and replace the cut with two cuts on structurally smaller formulae:

$$\frac{\dfrac{B \in B,\Delta \qquad B \supset C,\Gamma \Rightarrow \Delta,B}{\Gamma \Rightarrow \Delta,B}\,\text{invIL} \qquad \dfrac{\dfrac{B,\Gamma \overset{h_1-1}{\Rightarrow} \Delta,C,B \supset C \qquad A,\Gamma \overset{h_2}{\Rightarrow} \Delta}{B,\Gamma \Rightarrow \Delta,C}\,\text{r-cut} \qquad \dfrac{B \supset C,B,C,\Gamma \overset{h_2}{\Rightarrow} \Delta}{B,C,\Gamma \Rightarrow \Delta}\,\text{inv}}{B,\Gamma \Rightarrow \Delta}\,\text{scut}}{\Gamma \Rightarrow \Delta}\,\text{s-cut}$$

2. If $B \supset C \in \Delta$ we can just permute the cut upwards (i.e. do an r-cut), yielding the following derivation:

$$\dfrac{B \supset C \in \Gamma \qquad \dfrac{\dfrac{B,\Gamma \overset{h_1-1}{\Rightarrow}_D A,\Delta}{B,\Gamma \overset{h_1-1}{\Rightarrow}_D A,B,\Delta}\text{weak.} \qquad \dfrac{B,\Gamma \overset{h_2}{\Rightarrow}_D \Delta}{A,B,\Gamma \overset{h_2}{\Rightarrow} \Delta}\text{weak.}}{B,\Gamma \Rightarrow C,\Delta}\text{r-cut}}{\Gamma \Rightarrow \Delta}$$

modal rule: Just as in the intuitionistic case, we need a second case analysis on the derivation $A,\Gamma \overset{h_2}{\Rightarrow} \Delta$ when the modal rule was used to derive $\Gamma \overset{h_1}{\Rightarrow} A,\Delta$.

We take a look at the case when the modal rule is used in both derivations.

$$\dfrac{\dfrac{\Box A_0 \in \Delta, A \qquad \Box^-(\Gamma) \Rightarrow A_0}{\Gamma \Rightarrow \Delta, A} \qquad \dfrac{\Box A_1 \in \Delta \qquad \Box^-(A,\Gamma) \Rightarrow A_1}{A,\Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

We have two cases: Either $A = \Box A_0$ or $\Box A_0 \in \Delta$.

1. In the first case, we first use the modal and then do an s-cut:

$$\dfrac{\Box A_1 \in \Delta \qquad \dfrac{\dfrac{\Box^-(\Gamma) \Rightarrow A_0}{\Box^-(\Gamma) \Rightarrow A_0, A_1}\text{weak.} \qquad \dfrac{\Box^-(A,\Gamma) \Rightarrow A_1}{A_0, \Box^-(\Gamma) \Rightarrow A_1}\text{weak.}}{\Box^-(\Gamma) \Rightarrow A_1}\text{s-cut}}{\Gamma \Rightarrow \Delta}$$

2. If $\Box A_0 \in \Delta$, we can directly use the modal rule.

$$\dfrac{\Box A_0 \in \Delta \qquad \Box^-(\Delta) \Rightarrow A_0}{\Gamma \Rightarrow \Delta}$$

We can now prove the decidability of logic K using a similar proof search as in the intuitionistic case.

Theorem 6.23 *K is decidable.*

6.6 *Other approaches*

In this section, we compare our approach with other approaches to mechanizing cut-elimination-proofs. We will look at the different possible encodings of the sequent calculus and other possible strategies to prove cut admissible. Dang (2015) proves a different version of cut:

$$\dfrac{\Gamma \Rightarrow A,\Delta \qquad \Gamma' \Rightarrow \Delta'}{\Gamma,\Gamma' \setminus A \Rightarrow \Delta \setminus A,\Delta'}$$

According to Dang, a similar generalization was suggested by Girard (1989). In his Coq mechanization, he does not need a height-bounded system and does not prove any inversion results, but instead uses 3 nested inductions. We use a single, but special induction (with up to two inversions), but need to prove inversion lemmas beforehand.

Schepler (2016) uses the following generalization to prove admissibility of cut for intuitionistic propositional logic:

$$\dfrac{\Gamma \Rightarrow B \qquad \Gamma' \subseteq B,\Gamma \qquad \Gamma' \Rightarrow A}{\Gamma \Rightarrow A}$$

His proof is by subformula-induction on the cut-formula and induction on the derivation of $\Gamma' \Rightarrow A$ afterwards.

We tried producing a similar proof using one of these generalizations for IEL but ultimately failed. This is probably due to in both cases not being able to eliminate cuts on the same formula with a smaller height, though it could also be that with a clever trick, such a proof is possible. In terms of complexity (measured in lines of code), these approaches are fairly similar to our approach.

One difference between their and our approach is that we prove the context sharing ($\Gamma \Rightarrow A, \Delta \to A, \Gamma \Rightarrow \Delta \to \Gamma \Rightarrow \Delta$) instead of context disjoint ($\Gamma_1 \Rightarrow A, \Delta_1 \to A, \Gamma_2 \Rightarrow \Delta_2 \to \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$) versions of cut. We could have also proven context-disjoint versions, this makes a proof a little bit more complicated, however in any system with weakening, it does not make a difference which version we prove. Assume for example, we have proven context-sharing cut. We can then easily obtain context-disjoint cut:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow A, \Delta_1}{\Gamma_1, \Gamma_2 \Rightarrow A, \Delta_1}\text{weak.} \quad \dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{A, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\text{weak.}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\text{cs-cut}$$

That is, with context-sharing cut we can emulate the context-disjoint cut rule in a classical system. For the other direction, we use that $\Gamma \subseteq \Gamma, \Gamma$.

$$\dfrac{\dfrac{\Gamma \Rightarrow A, \Delta \quad A, \Gamma \Rightarrow \Delta}{\Gamma, \Gamma \Rightarrow \Delta, \Delta}\text{cd-cut}}{\Gamma \Rightarrow \Delta}\text{weak.}$$

Let us last discuss other possible ways to encode sequent systems in a proof assistant. One way to look at the different encodings is that all of them provide different answers to the question of how permutation invariant multisets can be represented using lists. As we will see all of these encodings are used in some mechanizations of proof-theoretic results.

A general observation we made was that there is a trade-off between representations which can be used to construct concrete derivations and being able to prove for example proof-theoretic results. For example there are representations, which prima facie seem very complicated and add a lot of mechanization overhead when actually constructing derivations which, that are well suited for proving cut-elimination (*e.g.* the permutation-representation). On the other hand, very simple encodings (*e.g.* the structural encoding) can make proofs of the high-level results much more tedious while being much better suited for actually constructing a concrete derivation. We will shortly mention benefits and drawbacks of each possible representation and showcase how the conjunction introduction rule would be encoded in the sequent calculus.

Membership encoding This is the encoding that we used for both K and IEL. It is elegant and gives short proofs, however it needs a special proof system similar to the GKI-calculus from Troelstra and Schwichtenberg (2000). The other representations we consider do not have such a strong restriction on the syntax of the inference rules but lead to more complicated proofs.

Explicit permutation rule The perhaps most straightforward encoding is to always have the principal formulas at the beginning of a list and add one rule allowing to permute lists arbitrarily in any derivation.

$$\frac{\Gamma \equiv_P \Gamma' \qquad \Gamma' \Rightarrow A}{\Omega \Rightarrow A}\ \text{Perm}$$

The main drawback of this encoding was that adding heights is complicated. Essentially one has to decide whether using the permutation rule shall increase the height of a derivation or not (i.e. the height stays the same). If it does increase the height, weakening is no longer depth preserving, if it does not increase the height proofs by induction on the derivation get more complicated (and induction on the height becomes infeasible). However, this encoding can ease mechanizing decidability proofs, since only the head of the list has to be matched on (e.g. Hara (2013)).

Permutation encoding This encoding makes the proofs very similar to textbook proofs. It incorporates the permutation rule into every rule of the sequent calculus, and thus there is no need to include a permutation exchange rule.

$$\frac{\Gamma \equiv_P B_1 \wedge B_2, \Gamma' \qquad \Gamma'' \equiv B_1, B_2, \Gamma' \qquad \Gamma'' \Rightarrow A}{\Gamma \Rightarrow A}$$

It seems to be easier to use such an encoding in proof assistants which have a lot of automation for dealing with permutations built-in. For example Michaelis and Nipkow (2017) prove cut-elimination using this approach, they claim that most cases in their cut-proof are solved by automation. At least in Coq, we were not able to prove these results using automation directly. However a cut-elimination proof using this encoding is possible, although the code is more complicated and proofs of all lemmata are more complicated. This is mainly due to worse automation for permutations in Coq. In every step when constructing a derivation, simple permutation equivalences need to be proven. We also cannot directly prove weakening with inclusion and instead need to prove both textbook weakening and textbook contraction to obtain the general weakening result. Using a permutation-solver[12] found on GitHub, which solves permutation equivalences like $A, \Gamma_1, D, \Gamma_2 \equiv_P A, B, \Gamma_2, \Gamma_1$ based on transforming permutation equivalences into equations over natural numbers, we were able to make the proofs feasible in Coq. However this is of course still a weaker kind of mechanization than what built-in automation for permutations might be able to provide. A lot of trivial steps in informal reasoning about permutations need to formalized.

[12] https://github.com/foreverbell/permutation-solver

Structural encoding In non-modal logics or more generally any logic were no rule modifies the complete context, it is possible to simulate multisets by squishing formulas in-between arbitrary lists.

$$\frac{\Gamma_1, B_1, B_2, \Gamma_2 \Rightarrow A}{\Gamma_1, B_1 \wedge B_2, \Gamma_2 \Rightarrow A}$$

It is an elegant mechanization, a possible drawback is that case analysis on the premisses are lengthy. This encoding was used by Penington (2018) and van Doorn (2015).

Overall evaluation Thus, for proving cut elimination for IEL (and IEL^-, K) only the permutation encoding and the system based on the G3I-calculus , we ultimately ended up using, can prove cut-elimination. We have cut-elimination proofs for both the representations, however the difference in code size and general elegance is profound: For the classical modal logic K[13] the proof with permutations is about 750 lines of Coq code. However this is only the main part of the proof, the complete proof depends on further on the permutation solver and general equivalence of textbook structural properties and general weakening (roughly 400 lines of code). In contrast to this, our cut-elimination proof utilizing the representation by Dang and Smolka is only 240 lines long. Many cases in the proof have a resemblance to the ones proven with the permutation encoding but are generally easier to solve, since no hard case analyses on permutation equivalence of lists are needed. Of course this too uses automation namely a lot of list automation from the Programming System Lab's Base Library,[14] for example, this helps Coq to solve list inclusions automatically. One advantage of the permutation encoding might be that the proofs are quite similar to the textbook proof, thus this suggests that many logics can be treated in a similar fashion (e.g. S4 and others).

[13] We only compare the results for K here, since here we prove context-sharing versions of cut for both representations, while for IEL the permutation-based encoding is used to prove context disjoint cut.

[14] https://github.com/uds-psl/base-library

In conclusion, while the permutation based-encoding is currently harder to mechanize in Coq, we believe that better library support and automation should make this encoding more feasible. The fact, that for example in Isabelle/HOL, which has built-in multiset support, there are more mechanizations of sequent calculi using such encodings (*e.g.* Cox (2019) and Michaelis and Nipkow (2017)) can be considered evidence for this. At the moment, however, the representation we ultimately use is a much better choice for mechanization. One of the benefits is that these representations can be directly used for the decidability proof. However, it is not obvious that such a representation can be found for any G3C or G3I-style sequent calculus, where it is much easier to argue this for the permutation based representation.

7
IEL and Epistemic Paradoxes

We first present the well-known Church-Fitch paradox (Fitch, 1963). It tries to show that the innocent assumption that every truth is knowable leads to an epistemic collapse and forces us to accept that every truth is known. We then discuss the IEL solution and shortly present the so-called typing approach. After discussing the IEL solution, we consider a second epistemic paradox, the paradox of idealization (Florio & Murzi, 2009). This paradox tries to establish the existence of an unknowable truth - which threatens the validity of IEL's co-reflection principle.

7.1 The Church-Fitch paradox

We have certain intuitions regarding the relationship between knowledge and truth. Fitch's paradox of knowability shows that two of these are incompatible. The paradox consists of a derivation, deriving the troubling omniscience claim, that every truth is known from the innocent assumption that every truth is knowable and the assumption that we are collectively non-omniscient. Note that the K-operator in this context is to be read as *somebody at sometime knows that P*.[1] Thus it presents a threat to any philosophical theory committed to the claim that every truth is possibly knowable, for example, verificationist theories like Dummets semantic anti-realism.[2]

We will refer to the principle that all truths are knowable as weak verificationism or the knowability principle and express it by

$$A \supset \diamond \mathsf{K}\, A. \tag{WVER}$$

The second intuition, which anti-realist theories usually share, is the claim that not every truth is known. That is, there is a true proposition A which is not known. The paradox consists in a derivation of a theorem which is often called omniscience from these assumptions and standard principles about knowledge.

$$A \supset \mathsf{K}\, A. \tag{OMN}$$

A rough proof sketch goes as follows: Assume p is an unknown truth. Thus by the knowability principle, it is possible to know $p \wedge \neg \mathsf{K}\, p$. But knowing such a conjunction is impossible since knowing

[1] There is discussion wether this how K should be interpreted for deriving the paradox. For example Flachs (2020, p.2) remarks that this will lead to knowledge losing the facticity and instead interprets it as *it is now known by someone*.

[2] As pointed out by Chung (2007), according to Kvanvig (2006) many philosophical theories (and not just verificationist theories of truth) from different areas are at least tacitly committed to the claim that all truths are knowable.

a conjunction implies knowing both conjuncts, *i.e.* $\mathsf{K}\, p$ and $\mathsf{K}\,\neg\mathsf{K}\, p$, however by facticity we know $\mathsf{K}\, p$ from the second conjunct and can derive a contradiction.

7.1.1 *Deriving Church-Fitch*

In this section, we will present the formal proof of the Church-Fitch-paradox. Our presentation follows the presentation in Wójcik (2020). In the original paper (Fitch, 1963) the presentation is slightly different, since Fitch does not use a K-modality directly but instead uses the concept of truth-classes where a truth-class α is a set s.t. $A \in \alpha \to A$. Fitch credits an anonymous referee for discovering the link between this paradox and knowledge, only fairly recently, the anonymous referee turned out to be Alonzo Church (Salerno, 2010). The re-discovery of the paradox (in an epistemic context) is often attributed to Hart (Bell & Hart, 1979).

We will begin by laying out the assumptions which are assumed in the derivation of the paradox. First, we need the rule of necessitation, which states that a theorem is necessary at any possible world:

$$\vdash A \implies \vdash \Box A \tag{Nec}$$

The second standard assumption regarding the modal logic expresses a relationship between possibility and necessity:

$$\neg\Diamond A \dashv\vdash \Box\neg A \tag{interdef}$$

The symbol $\dashv\vdash$ expresses inter-definability. While some instances of inter-definability are intuitionistically invalid (or should not be endorsed, see below), we argue that the specific principle used here is valid. For this recall the translation from modal into first-order logic: $\neg\Diamond A$ can be expressed as $\neg\exists w'.w' \Vdash A$ and $\Box\neg A$ as $\forall w'.w' \Vdash \neg A$. We can give a constructive proof of this equivalence, therefore this inter-definability principle is acceptable in intuitionistic logic. As De Vidi and Solomon (2001) argue other classically equivalent inter-definability laws behave strangely in an intuitionistic context. Using both $\Diamond A \coloneqq \neg\Box\neg A$ and $\Box A \coloneqq \neg\Diamond\neg A$ would allow deriving double-negation elimination for modal sentences, i.e. $\Box A \leftrightarrow \neg\neg\Box A$. It seems to be unplausible to reject double negation elimination for general sentences and only allow it for modal sentences, thus this stronger inter-definability is not intuitionistically acceptable. Now we come to the assumptions regarding K. As stated earlier, we assume the facticity of K.

$$KA \supset A \tag{Fact}$$

The last assumptions we need are distribution of K over conjunction and the non-omniscience assumption.

$$\mathsf{K}\,(A \wedge B) \supset \mathsf{K}\, A \wedge \mathsf{K}\, B \tag{Distr}$$

$$\exists A. A \wedge \neg\mathsf{K}\, A \tag{Non-Omn}$$

We can now formally derive the Church-Fitch paradox. First, we show that $\neg\mathsf{K}\,(A \land \neg\mathsf{K}\,A)$ for any A. So assume $\mathsf{K}\,(A \land \neg\mathsf{K}\,A)$. By distribution over conjunction one obtains $\mathsf{K}\,A \land \mathsf{K}\,\neg\mathsf{K}\,A$. By using facticity on the second assumption we have $\mathsf{K}\,A \land \neg\mathsf{K}\,A$ - which leads to a contradiction. Thus we have proven $\neg\mathsf{K}\,(A \land \neg\mathsf{K}\,A)$. By necessitation, we obtain $\Box\neg\mathsf{K}\,(A \land \neg\mathsf{K}\,A)$ and by inter-definability of the modal operators, we get $\neg\Diamond\,\mathsf{K}\,(A \land \neg\mathsf{K}\,A)$. But since we assumed $A \land \neg\mathsf{K}\,A$ we get $\Diamond\mathsf{K}\,(A \land \neg\mathsf{K}\,A)$ by the knowability principle, which obviously contradicts our assumption.

We showcase a Hilbert-style derivation of the paradox:

1.$\exists A.A \land \neg\mathsf{K}\,A$	(ass.)
2.$A \land \neg\mathsf{K}\,A$	from (1)
3.$\forall A.A \to \Diamond\mathsf{K}\,A$	(ass.)
4.$(A \land \mathsf{K}\,A) \to \Diamond\mathsf{K}\,(A \land \mathsf{K}\,A)$	Instance of (3)
5.$\Diamond\,\mathsf{K}\,(A \land \mathsf{K}\,A)$	(MP)
6.$\mathsf{K}\,(A \land \mathsf{K}\,A)$	(assumption for reductio)
7.$\mathsf{K}\,A \land \mathsf{K}\,\neg\mathsf{K}\,A$	(Distr)
8.$\mathsf{K}\,A \land \neg\mathsf{K}\,A$	(Fact)
9.$\neg\mathsf{K}\,(A \land \mathsf{K}\,A)$	
10.$\Box\,\neg\mathsf{K}\,(A \land \mathsf{K}\,A)$	(Nec)
11.$\neg\,\Diamond\,\mathsf{K}\,(A \land \mathsf{K}\,A)$	(interdef)

Since (11) contradicts (5), our non-omniscience assumption (1) is contradicted. Thus $\neg\exists A.\,(A \land \neg\mathsf{K}\,A)$, which is classically equivalent to $\forall A.\,A \supset \mathsf{K}\,A$. In intuitionistic logic, we are only committed to a weaker conclusion, namely $\forall A.\,\mathsf{K}\,A \supset \neg\neg A$ (and of course other intuitionistically equivalent theorems). There are other ways to derive the paradox, for example, Maffezioli et al. (2013) present a derivation in a sequent calculus.

7.1.2 *Overview of Solution Attempts*

Strategies to block the paradox can be roughly classified into the following categories (this division is based on Brogaard and Salerno (2019)):

- Syntactic Restriction strategies: These strategies try to restrict the sets of formulas to which K can be applied. Syntactic restriction thus introduce a logical property F and restrict the knowability principle in the following way (Brogaard & Salerno, 2019):

$$A \to \Diamond\mathsf{K}\,A \text{ for all A such that } F\,A \qquad \text{(F-KP)}$$

  Most notable here is Tennant's (1977) proposal to restrict weak verificationism to *Cartesian* propositions, where a proposition A is a cartesian if $\mathsf{K}\,A$ is consistent. Another commonly discussed approach is Dummett's proposal to restrict knowability to so-called *basic* statements. These strategies are often criticized to be *ad-hoc*, since there seems to be no motivation for introducing these constraints other than blocking the derivation of the paradox.

- Logical revisions: These approaches try to solve the paradox by changing the underlying logic. Most notable here are approaches using intuitionistic or para-consistent logics.[3]

- Others: There are authors who, for example, suggest that knowability is not distributing over conjunction. However versions of the paradox can be run, even when distribution over conjunction is rejected (e.g.Jago (2010)).

[3] Para-consistent logics do not have an explosion rule, thus both $\mathsf{K}\,A$ and $\neg\mathsf{K}\,A$ can hold without any theorem being derivable.

There also is some general discussion, whether $A \to \Diamond\mathsf{K}\,A$ is an adequate expression for the knowability principle. Especially troubling here is using $\Diamond\mathsf{K}\,A$ for expressing knowability. For example Jarmużek et al. (2021) criticize that knowability is a *de re* modality but is expressed as a *de dicto* modality in the paradox. To see this consider the difference between the sentences:

a) It is possible that John knows that a Ferrari is a sports car. (De dicto)

b) John may know that a Ferrari is a sports car. (De re)

They argue that the second captures the intended sense behind the idea that all truths are knowable more closely. Someone who asserts this wants to assert that it is possible to know a truth in the actual world and not that there is a possible world in which it is known.

Fuhrmann (2014) makes a similar point by distinguishing between *possible* and potential knowledge; here possible knowledge is knowledge at a possible world and potential knowledge represents the possibility of knowing something in the actual world.[4]

[4] This is realized by mixing Kripke-models with belief revision.

Furthermore, Costa-Leite (2006) argues that knowability is factive i.e. if A is knowable, A is true. If this was formalized as $\Diamond\mathsf{K}\,A \supset A$, it would allow to derive $A \leftrightarrow \Diamond\mathsf{K}\,A$, which would equate knowability and truth.

7.1.3 The Intuitionistic Response

There have been arguments brought forward that just adopting intuitionistic logic will resolve the paradox. Using intuitionistic logic, we cannot obtain general omniscience but only the following principle, which is what we know as the intuitionistic reflection principle:

$$A \supset \neg\neg\mathsf{K}\,A$$

Williamson interprets this as expressing that once A is known, it can no longer be falsified:

> It forbids intuitionists to produce claimed *instances* of truth that will never be known. (Williamson, 1982, p. 206)

Thus it is a weaker result than omniscience. If intuitionistic negation is read as *it is impossible to refute A*, it only states that once knowledge is acquired, it can not be falsified. Williamson further argues that the intuitionist could still happily accept

$$\neg\neg(\forall A.\, A \supset \mathsf{K}\, A)$$

since it does not contradict

$$\neg\exists A.\, A.\, \wedge \neg\mathsf{K}\, A.$$

Dummett (2009) comes to a similar conclusion and urges us to accept $A \supset \neg\neg\mathsf{K}\, A$ as the knowability principle instead. Percival (1990) identifies three conclusions from the intuitionistic response. These are especially relevant since these can also be viewed as a criticism of IELs intuitionistic reflection principle. His analysis basically consists of 3 theorems, he argues proponents of intuitionistic solutions to the paradox are committed to and an analysis, why each is troublesome.

Lemma 7.1 *Intuitionist theories of knowledge, who solve the paradox by endorsing $A \supset \neg\neg\mathsf{K}\, A$ are committed to the following 3 theorems:*

(i) $\neg\mathsf{K}\, A \supset \neg A$

(ii) $\neg\mathsf{K}\, A \leftrightarrow \neg A$

(iii) $\neg(\neg\mathsf{K}\, A \wedge \neg\mathsf{K}\, \neg A)$

Proof. We derive each seperately:

(i) For deriving the first statement, we first use the contraposition law on intuitionistic reflection and obtain $\neg\neg\neg\mathsf{K}\, A \supset \neg A$. Since $\neg A \supset \neg\neg\neg A$ is valid in intuitionistic logic, we obtain the desired result.

(ii) Is obtained using the classical truth condition and (i).

(iii) Assume $\neg\mathsf{K}\, A$ and $\neg\mathsf{K}\, \neg A$. Using intuitionistic reflection on both hypotheses gives us $\neg A$ and $\neg\neg A$ which allows us to derive a contradiction □

Note, that his criticism of these theorems also is a criticism of IEL principles, as IEL is committed to all of the above.

Lemma 7.2 *In IEL, all three theorems from above can be derived.*

Proof. We only show how to derive $\neg(\neg\mathsf{K}\, A \wedge \neg\mathsf{K}\, \neg A)$. To derive a contradiction, assume $\neg\mathsf{K}\, A \wedge \neg\mathsf{K}\, \neg A$. Thus (by conjunction elimination) we have proofs of $\neg\mathsf{K}\, A$ and $\neg\mathsf{K}\, \neg A$ and need to prove falsity. Using the second hypothesis, we need to show $\mathsf{K}\, \neg A$. By co-reflection it suffices to show $\neg A$, thus we have A as an additional assumption. Now we use the first conjunct and need to prove $\mathsf{K}\, A$, again by co-reflection it suffices to prove A, but we have a hypothesis which is a proof of A. The other proofs can be found in the Coq development. □

Let us now consider Percivals arguments against the logical equivalence of $\neg A$ and $\neg\mathsf{K}\, A$. He asks us to consider a mathematical statement

$\neg A$. He reads $\neg \mathsf{K}\, A$ as the impossibility that A is known and $\neg A$ as A being false. Now, according to him $\neg \mathsf{K}\, A$ is contingent while $\neg A$ is not.[5] But it is hard to imagine that only one of two logically equivalent propositions is contingent.

[5] A proposition p is contingent if it is not necessarily true, formally: cont $p := \Diamond p \wedge \Diamond \neg p$.

Lastly, he argues that we can envision a situation in which our knowledge of a necessary a posteriori truth P is dependent on a contingent truth (De Vidi & Solomon, 2001):

> For example, imagine a laboratory researcher about to record the litter of Doris rabbit who turns around to see that Mabel rabbit shifts her offspring Peter into Doris's nest. He then thinks: if I hadn't looked around it would never have been known that Mabel is Peter's parent. (Percival, 1990, p. 184-185)

In the example, P is the proposition that Mabel is not Peter's parent. In the counterfactual situation, where the researcher had not turned around in the right moment, P would not have been discovered and thus not known by the researcher, thus *the researcher had turned around* is a contingent truth on which the knowledge is dependent. Therefore, in the counterfactual situation where the researcher had not turned around, according to Percival, both P and $\neg \mathsf{K}\, P$ would be true, refuting the logical equivalence.

The third consequence $\neg(\neg \mathsf{K}\, A \wedge \neg \mathsf{K}\, \neg A)$ is interpreted by Percival as stating that no statement is forever undecided,[6] which would be just as troubling as an omniscience claim.

[6] Here, a proposition A is called *undecided* if neither A nor $\neg A$ is known. Thus it is not the computer-scientific meaning of the term.

We also consider objections raised by Artemov and Protopopescu (2016) and De Vidi and Solomon (2001). De Vidi and Solomon argue that Percival essentially uses a classical interpretation of the logical constants and misses that intuitionistic negation is in some sense an *impossibility* operator, since if $\neg A$ is semantically valid, A is false at any successor state. Thus $\neg A$ does not express currently not knowing A but impossibility to know A.

In the researcher-mouse example Percival asserts $\neg \mathsf{K}\, P$ since it will never be known that Mabel is Peters parent. However, while it may not be known in the counterfactual situation at the moment, it would in theory, for example through methods as DNA-testing be possible to come to know P; thus there is a possible extension of the knowledge where P is known.

His last example also bases on the false reading of $\neg \mathsf{K}\, A$. If it were the case that the correct reading would be that no statement would forever remain undecided the argument would be perfectly fine. However, a better translation might be that it is impossible that both P and $\neg P$ are unknowable. That the validity of $\neg(\neg \mathsf{K}\, A \wedge \neg \mathsf{K}\, \neg A)$ intuitionistically does not force every proposition to be decided can also be seen by considering a Kripke-model where every world w_i has one successor where P is forced and one where P is not forced.

Artemov and Protopopescu argue in a similar fashion that each of Percivals conclusions is perfectly acceptable when read under an IEL conception of knowledge. $\neg \mathsf{K}\, A \leftrightarrow \neg A$ only states that if A cannot be proven it is impossible to verify that A and vice-versa, both are acceptable ideas. If any truth can be verified and it is provable that a verification of A is impossible, A cannot be proven (since else it would be verifiable).

7.1.4 Typing K

Some recent (e.g. Linsky (2010), Paseau (2008), and Raclavský (2018)) articles on the paradox try to restrict it by typing the K-modality. The general idea of all typing approaches is to stratify the knowledge predicate into an infinite hierarchy of K-modalities (akin to universes levels in constructive type theories). Propositions without any intensional operators (in our case the knowledge operator) are level 1 propositions.

For example, the well known socrates paradox, *I know, that I don't know anything* can be resolved with a typing approach. When Socrates claims that he knows nothing, he asserts a second order proposition

$$\mathsf{K}^2(\forall p^1. \neg \mathsf{K}^1 p)$$

which is no longer contradictory since he only claims not to know any 1st-order proposition but this knowledge is a second-order proposition. Similarly the derivation of the Church-Fitch paradox can also be blocked. Essentially with distribution one only obtains

$$\mathsf{K}^2 p \wedge \neg \mathsf{K}^1 p$$

in step 8, which no longer is contradictory. Typing approaches, just as syntactic restriction strategies, have been criticized to be an ad-hoc solution, recently by Carrara and Fassio (2011) who argue that there can be no paradox-independent motivation for introducing types, however Raclavský (2018) argues against this. Others object that decomposing the knowability-predicate is poorly motivated since nothing in ordinary language would suggest this. Paseau (2008) (among a general discussion of a typing approaches) criticizes this line of response by comparing the motivation for typing knowledge to the motivation to typing truth.

7.1.5 IEL Response

The IEL response, as given by Artemov and Protopopescu (2016), is that the paradox, as a threat to intuitionistic theories of knowledge does not exist. Since it critically rests on a specific understanding of certain (modal) formulas. In this their approach is similar to for example criticizing $A \supset \Diamond \mathsf{K} A$ as not expressing knowability.

According to them the paradox critically rests on the following 3 assumptions:

- $A \supset \mathsf{K} A$ means all truths are known.
- $A \supset \Diamond \mathsf{K} A$ means that all truths are knowable
- That all truths are knowable is definite of intuitionistic truth.

However since all of the formulas meanings are different in IEL and since the (classically objectionable) conclusion $A \supset \mathsf{K} A$ is a defining principle of knowledge in IEL, there is no paradox. Let us recapitulate, how Artemov and Protopopescu read the formulas: Against the first reading they object that it just expresses the constructive nature of intuitionistic truth, that is that any verified proposition is known, thus it is not an omniscience claim. Against the second principle, they argue that if it read as a classical principle it is plain wrong, since not every classical truth is knowable. They argue, that as an intuitionistic principle it is strictly weaker than the acceptable co-reflection principle. They admit it might be suitable under a reading of proofs as timeless platonic entities combined with a reading of $\mathsf{K}\,A$ as A is actually known could provide reasonable semantics for the knowability principle. It can then be read as, if a proposition has a proof (in a platonist, timeless sense) it can actually be known. The third principle is also misguided: the constructive nature and not the knowability of any truth is essential for an intuitionistic understanding of knowledge. Their argument against Church-Fitch now is, that the conclusion is perfectly acceptable in IEL, thus there is no need to even discuss if the derivation is possible in intuitionistic epistemic logic.

7.1.6 Discussion

One objection against the IEL response is that intuitionistic knowability only extends to mathematical propositions since non-mathematical statements are not susceptible to notions as proof. A similar objection is already raised by Percival. Artemov and Protopopescu claim that BHK is universally applicable since it makes no reference to mathematical concepts such as numbers, functions, etc. and only relies on notions as justification or concrete evidence, notions which have always, and especially after Gettier (1963), influential to epistemology. Furthermore notions such as proof and conclusive evidence are also used in non-mathematical settings for example in the context of legal standards or arguments in a court of law.

While I do not believe that this is necessarily the best argument for the claim, I would agree that notions such as verification can also extend to empirical propositions. However in combination with BHK-semantics this will surely lead to a vastly different account of knowledge. I will try to illustrate this with two examples from the literature, one concerned with the BHK-interpretation of disjunctions and another one about negations.

Edgington (1981) points out that the BHK-interpreation of disjunction will limit the disjunctions we can assert, since asserting a disjunction $A \vee B$ is only possible if one either verified that A or verified that B. Consider a factory worker sorting balls by color. He sorts balls which are either blue or green into one bag and all other balls into another bag. He now might remember that the 5-th ball he sorted was put into the first bag but does not remember which color the ball was. Since he does not remember wether the ball was blue or green, he cannot know the disjunction *the 5th ball was blue or the* 5*-th ball was green*, since he would need to know which color the ball had. While this is already disconcerting, we can imagine more consequences: Consider a language very similar to English with the only difference that the language has a special word *bleen* for objects which are either blue or green. We now have two options, either we accept that the ball is bleen or deny it. If we accept it, it seems that inventing new words can circumvent the BHK-semantics and that many words we take to have the same meaning as for example *sibling* or *brother or sister* can no longer be used interchangeably. If we deny it, we would also have to accept a similar division of predicates in other cases, for example of bleen into a more gray or less bleen, which is only a troublesome consequence, this might it doesn't seem apparent when we can stop such a division of predicates. For example *green* could be divided into *ligher green* or *darker green* (and lighter green could again be divided further).

De (2013) argues that BHK-semantics need to be enriched with a new operator to model empirical negation. He tries to show that intuitionistic negation cannot be used to express some empirical statements. One example he considers is the proposition that the Goldbach-conjecture is currently undecided. If we use intuitionistic negation to express this, we could do this as $\neg(G \vee \neg G)$. By the definition of intuitionistic negation this is equivalent to $(G \vee \neg G) \to \bot$, However this is too strong: It models the fact that G can never be decided and not the fact that the statement is currently not decided. Thus a new operator is needed, which he tries to construct semantics for.

While both examples are not enough to defeat the project of applying BHK-semantics to empirical propositions it shows that there are at least some questions to be answered if we want to combine BHK-semantics and empirical knowledge and defenders of this view owe us a better explanation than just referring to Gettier and the fact that there is mention of mathematical objects in a BHK-proof.

However, even if we ignore those arguments for now and assume that IEL can also be used to knowledge about empirical statements, the question remains wether this approach can solve the paradox. I would argue that it solves the paradox for intuitionistic knowledge, however not for classical knowledge. If we take the Church-Fitch paradox to be an argument that two intuitions regarding classical knowledge (or rather knowledge as we often use it) are inconsistent, it is not resolved by the IEL solution. Note, that there is a difference here between the intuitionistic solution and the IEL solution, as only the latter basically changes the meaning of the K-operator. But changing the meaning of the K-operator changes the argument. For example consider a mathematician who, by some reasoning, came to the apparently false conclusion that $2 - 2 \neq 0$. If we would take the IEL-response as principled, we would have to accept that suggesting to just interpret the minus as plus would resolve the mathematicians paradox. That is to say, the paradox as a threat to classical knowledge is not resolved.

Secondly, it seems to be hard to capture Church-Fitch like arguments in IEL. IEL seems to be too inexpressive to model knowability, since first there is no way to distinguish between the existence of an intuitionistic proof and having cognitive access to it and secondly IEL seems to model an individual's rather than a collective's knowledge, since otherwise the co-reflection principle is not well unjustified. However even intuitionistic knowledge will have to be committed to a knowability-like thesis akin to "If agent is in possession but has no cognitive access to a proof of p, he might know that p". If being in possession of an intuitionistic proof is factive it might be possible to run a version of the paradox with this operator. Of course, the IEL strategy would be to claim that the operation is not factive because the truth (in the sense that the agent can assert A) is only established once the agent has realized that he possesses a proof.

Thus the IEL response does not successfully resolves the Church-Fitch paradox for classical knowledge.

7.2 *Paradox of Idealization*

In this section we will first state the *paradox of idealization*, as introduced in Florio and Murzi (2009) and then apply it to IEL. The argument shows that under certain assumptions, intuitionists are committed to the existence of unknowable truths. We argue that the paradox of idealization can be applied to IEL, which conceptually threatens the co-reflection principle. If this is the case, IEL endorsers have no option but to bite the bullet and accept strict finitism (that is roughly speaking denying any idealization). However, while strict finitism itself already has "highly revisionary consequences" (Florio & Murzi, 2009) we will see that it does not work well in tandem with co-reflection.

The paradox of idealization can be derived in a multimodal multi-agent epistemic logic, with operators **K**, $\Diamond$, and $\Box$, where as usual the diamond is interpreted as possibility, the box as necessity, and the K as a knowledge modality. Just as in the paradox of knowability (Fitch, 1963), the assumptions on the rules governing K are quite modest e.g. only distribution over conjunction, facticity, and distribution (over implication) are necessary. We start by analyzing the principle of weak verificationism

$$A \supset \Diamond \mathsf{K}\, A \tag{WVER}$$

which states that every truth can be known. It is weaker than IELs co-reflection principle. Using this principle, it is easy to introduce strict finitism and motivate the need for idealization. Strict finitists will claim that the K in the formula above, expresses knowability by real agents. That is any truth can in principle be known by agents just like us. Most anti-realists would reject this and reply that the above formula expresses knowability in an idealized sense. That is, every truth is knowable by agents just like us or agents whose cognitive capacities finitely exceed ours (Murzi, 2010).

There are good reasons, why most anti-realists reject strict finitism. It seems unreasonable that propositions cannot be true or known for mere medical limitations. Intuitionists routinely assert propositions ranging over all natural numbers, for example, the existence of a computable primality test can be used to assert that every number is a prime or composite: $\forall n.\, p(n) \vee \neg p(n)$.[7] Let us further assume that the proof sizes of $p(n) \vee \neg p(n)$ grow linearly.[8] Call a number n apodictic (Dummett, 1975) if proofs of length n can be checked in practice. For example, 100 is clearly an apodictic number but Gogol (10^{100}) is not (Magidor, 2012; Montesano Montessori, 2019). Our intuition is that even for an apodictic number a, we should be able to know $p(a) \vee \neg p(a)$, even though we will not be able to check the proof. Thus if anti-realists still accept the principle WVER for this specific proposition, they are committed to an idealized account of knowability. Note that in theory any decidable proposition whose smallest proof size is an apodictic number can be used to establish that knowability in WVER is knowability in an idealized sense (unless of course, such propositions are not seen as true).

[7] We assume that p is the primality predicate.
[8] A monotonously increasing proof complexity would also suffice but make the argument slightly more involved.

7.2.1 *Deriving the Paradox*

The first premise of the idealization paradox tries to capture the requirement for idealization. This is expressed using a predicate $I(a)$ on agents, where $I(a)$ expresses, that the agent a is idealized that is the agents' cognitive capacities can only finitely exceed ours.

Our first premise frames the requirement for idealization as an existential. There is a proposition that can only be known by idealized agents:

$$\exists P. P \wedge \Box \forall a. \mathsf{K}_a(P) \supset I(x). \tag{P_1}$$

Note, that this is quite a strong premise: There is a single proposition P which can only be known by idealized agents in any possible world. The second assumption is that no idealized agents exist. This seems to be a well-motivated thesis because idealized agents are exactly those agents whose cognitive capacities exceed any actual agents. If we take actual agents to mean existing agents, it is an analytical truth.

$$\forall x. \neg I(x) \tag{P_2}$$

These are all the assumptions we need, apart from standard epistemic assumptions (*i.e.* knowledge is factive and distributes over conjunction). Now consider the following proposition , which will be our unknowable truth:

$$P \wedge \forall x.\, \neg I(x). \tag{7.1}$$

It is true (since both conjuncts have a proof), however, it cannot be known. Intuitively this is the case since the left conjunct can only be known by idealized agents, whose existence is impossible (because of the right conjunct).

Lemma 7.3 *Equation (7.1) cannot be known.*

Proof. Assume the proposition is known by an agent a. Since knowledge distributes over conjunction, we obtain $\mathsf{K}_a(P) \wedge \mathsf{K}_a \forall x.\, \neg I(x)$. Using P_1 we obtain $I(a)$, using facticity and distributivity of knowledge we obtain $\forall x.\, \neg I(x)$ - a contradiction. □

That already is the paradox: from rejecting strict finitism and some innocent assumptions about the knowledge operator we obtained the existence of an unknown truth.

As Florio and Murzi (2009) argue, this can be generalized to any formula A and $P(x, A)$ s.t.

$$\exists A.A \wedge \Box(\forall a.\mathsf{K}_a(A) \supset P(x, A) \wedge \neg\exists x.P(x, A)),$$

even the well-known Church-Fitch knowability paradox (Fitch, 1963) can be seen as an instance of it (where $P(x, A) = \mathsf{K}_x(A)$).

Replies Let us consider one objection presented in Florio and Murzi (2009). The objection consists in claiming that any truth can be known in at least one possible world *i.e.* while there may be truths which cannot be known in a concrete possible world, there always is a different possible world where it can be known by a non-idealized agent. One possible way to motivate this thesis is that other possible worlds might use a different language, which is better suited for proving a specific proposition. This objection formally consists in endorsing the following principle:

$$\forall A.\,(A \supset \Diamond\exists x(K_x(A) \wedge \neg I(x)). \tag{7.2}$$

Florio and Murzi claim that by introducing a proposition s, which forces the spatial structure of a world to be similar to the actual world this objection can be blocked. Thus let s be a description of the actual world (which for example forces the spacial time structure or the language) [9], now consider

$$\exists A.A \wedge s \wedge \Box(\forall x.\,\mathsf{K}_x(A \wedge s) \supset I(x))). \tag{7.3}$$

[9] Essentially s shall force the minimum proof size of A to be larger than the agents cognitive capacities.

The argument now runs as before, additionally using that the world where the agent knows $A \wedge s$ must additionally have the same spacial time structure as the actual world since this is forced by s.

Raclavský (2018) criticizes that the paradox of idealization is based on the obviously self-refuting assumption that an idealized agent knows that no idealized agents exist. Thus it is not necessary to even consider typing approaches to reject this inference. We would argue that this is not an invalid inference but at the core of how the unknowable truth has been constructed.

7.2.2 *Discussion*

Now, does the paradox of idealization threaten IEL or is it a refutation of the co-reflection principle? First, there is the technical difficulty that the knowledge paradox uses a multi-agent logic. At first glance, the IEL principles seem hard to apply in a multi-agent logic. If the co-reflection principle is adapted naively as $A \to \mathsf{K}_a(A)$ for every agent a, the aforementioned justification of the co-reflection principle would not work anymore and any agent would know all theorems of intuitionistic propositional calculus and not just those for which she has access to a proof. However, the intuitionistic and distributivity principles seem to translate easily.[10] So can $P \wedge \forall x.I(x)$ be known under an IEL-interpretation?

Lemma 7.4 *Equation (7.1) cannot be known, when using IEL principles.*

Proof. Again, assume the proposition is known by an agent a. Since knowledge distributes over conjunction we obtain $\mathsf{K}_a(P) \wedge \mathsf{K}_a(\forall x.\neg I(x))$. Again from (P1) we can derive $I(a)$. Now we cannot use reflection, but intuitionistic reflection and obtain $\neg\neg\forall x.Ix$, which still is inconsistent with $I(a)$. □

Note, that this argument does not even depend on how IELs co-reflection principle is expressed in a multi-agent setting. We could try to express co-reflection as

$$\forall A.\, A \supset \forall a.\, \diamond \mathsf{K}_a(A) \tag{7.4}$$

or as

$$\forall A.A \supset \exists a.\, \diamond \mathsf{K}_a(A). \tag{7.5}$$

The important question here is if the agents (over which the universal quantification ranges) can be idealized or not. If it is possible that A can only be known by idealized agents, we can derive the paradox of idealization. Otherwise, it is possible to bite the bullet and endorse strict finitism in IEL. At least for IEL this seems to be the better and maybe more natural strategy, since the idea behind the co-reflection principle already relies on the agent being able to verify a proof of A. It would limit knowledge to what can actually be verified *e.g.* it would not be possible for an agent to know decidable propositions involving apodictic numbers, even if there is a for example a verified computer program that can construct a proof for any n.

While even this might be acceptable, though severely limiting the propositions which are true and can be known, it is questionable if the co-reflection axiom can work well with strict finitism. Consider an agent constructing a sequence of proofs A, $\mathsf{K}\,A$, $\mathsf{K}\,\mathsf{K}\,A$, ... If these proofs increase in size and there is a limit on her cognitive capacities, there will be an n s.t. the agent can know $\mathsf{K}^n(A)$ but not $\mathsf{K}^{n+1}(A)$. This argument can also be framed as a *happy sorites argument*[11], consider the following framing:

A can be known
For any A: If A can be known, so can $\mathsf{K}\,A$.

$\mathsf{K}^{2^{100}}(A)$ can be known ∴

This would indeed make the co-reflection principle unplausible. It would contradict the well-motivated assumption that we actually cannot know propositions with proof sizes of 2^{100}. However, there are some arguments that strict finitism and the existence of happy sets (*i.e.* sets which are closed under successor but bounded) are not inconsistent (Magidor, 2012), for example one could argue that proofs by inductions are not (or only in a restricted sense) available to the strict finitist.

[10] For example the intuitionistic reflection principle can be translated as $\forall a A.\, \mathsf{K}_a(A) \supset \neg\neg A$.

[11] Montesano Montessori (2019) calls an argument a happy sorites argument if all the premisses are true but the conclusion obviously is not.

Remarks The paradox of idealization has not generated a lot follow-up scholarly work. We are aware of Akcelik (2016) claiming to solve the paradox. He achieves this by restricting the scope of $\forall a. \neg I(x)$ to actual agents.

8 Conclusion

We have presented results about IEL in a constructive setting. Our main results included a constructive decidability proof and a constructive proof of completeness and strong quasi-completeness for modified semantics. As a second result, we have studied and compared approaches to mechanizing sequent calculi and especially cut-elimination proofs in the Coq proof assistant. We formulated sequent calculi which were well suited for mechanization and were able to extend the results to the classical modal logic K.

In this chapter we discuss related and future work.

8.1 Related Work

This section is split into three subsections: One concerned with papers about IEL, which do not necessarily have a mechanization component, and two others concerned with (mostly) mechanized completeness and one about mechanized cut-elimination (and decidability) proofs, respectively.

IEL Of course the main reference for IEL is the paper introducing the logic by Artemov and Protopopescu (2016). Protopopescu (2016a) furthermore proves soundness and completeness of embeddings from IEL to S4. His dissertation (Protopopescu, 2016b) consists of two more papers on IEL, one investigating the connection between IEL and modal logics of verifications and one about fallibilistic knowledge.

The proof theory of IEL has been studied by Krupski and Yatmanov (2016). Su and Sano (2019) proposed a sequent calculus for IEL with the subformula property and extended IEL with first order quantification. As already mentioned, their calculus for IEL uses sets of formulas with at most 1 element in the succedent of some rules. Fiorino (2021) has further studied proof search for IEL. Tarau (2019) develops a theorem prover for IEL using Prolog and presents embeddings from IEL into IPC (intuitionistic propositional calculus), however soundness or completeness proofs about the embeddings are not given. We tried to investigate those, but were only able to formally verify a soundness proof. This allows for proving consistency of IEL utilizing the consistency of IPC (Hagemeier, 2020). Recently it has been suggested by de Groot et al. (2021) that IEL's modality can be embedded into Heyting-Lewis logic with a strict-implication modality.

There have been some type-theoretic investigations into IEL and IEL^-. Perini Brogi (2021) develops a typed modal λ-calculus which is Curry-Howard-equivalent to IEL. One very interesting result of this work is that the knowledge modality in IEL seems to be weaker than type truncation; thus simply reading K as type truncation (or inhabitedness) seems to be too simple. Furthermore Rogozin (2021) proposes a modal λ-calculus which is Curry-Howard-equivalent to IEL^-.

Formalized completeness of modal logics Bentzen (2019) proves completeness and soundness of the classical modal logic S_5 in Lean. His proofs are entirely classical. There are slightly different Henkin-style constructions, for example van Dalen (2013) uses a construction where only disjunctive formulas are enumerated and one of the disjuncts is added to the context. It might be interesting to formalize these. Doczkal and Smolka (2016) prove completeness and decidability for CTL in Coq/SSreflect.

Cut-elimination and decidability The main references for our decidability and cut-elimination proofs are the report by Dang (2015) and lecture notes by Smolka and Brown (2012). Michaelis and Nipkow (2017) use a similar method as we do, to prove (among other results like completeness) cut-elimination for classical propositional logic. They formalize their results in the Isabelle/Hol. Park (2013) formalizes a bidirectional decision procedure for an intuitionistic modal logic using Coq. Hara (2013) formalizes decidability of intuitionistic propositional logic in Coq. The decision procedure uses a terminating search in a cut-free sequent calculus. The result is obtained by induction on the multiset ordering. Chaudhuri et al. (2017) prove cut-elimination for linear logic in Abella. Their encoding is similar to the permutation encoding in Coq, they also mention to have a Coq version of their code. Van Doorn (2015) proves soundness, completeness and cut-elimination for classical propositional logic in Coq, his cut-elimination proof is semantic. Penington (2018) has an incomplete mechanization of cut elimination following Troelstra and Schwichtenberg in Coq. Cox (2019) proves equivalence between a sequent calculus and natural deduction for intuitionistic propositional logic in Isabelle/Hol. Indrzejczak (2017) proposes to prove tautology elimination instead of cut-elimination. He claims that the resulting proofs are easier. Wu and Goré (2019) claim to mechanize the first verified decision procedures for K, KT and S4. They formalize their results in Lean und use a tableau-method.

8.2 *Future Work*

While our decidability procedure is correct, it is currently not very efficient. A first step to this end, would be to mechanize the decision procedure used for obtaining PSPACE-completeness by Krupski and

Yatmanov. Our current procedure is needs more than polynomial space since the number of subformulas is not polynomial in the size of a formula. It might also be worthwhile to verify the algorithms given in Fiorino (2021), who give a sequent representation for IEL, with linear sized proof trees. Their refutation calculus could also be used to obtain an informative decider for IEL, e.g. a decision procedure that constructs either a derivation or , if $\Gamma \nvdash A$, a countermodel.

It would certainly be interesting to further if the additional axioms used in the strong-completeness proof can be weakened or to establish which classical axioms can be proven when assuming . For example, similar analyses (among other work) were done by Forster et al. (2021), who investigated first-order intuitionistic logic.

Another possibly interesting area for future research, would be to extend IEL to multiple agents, similar to Jäger and Marti (2016), we already outlined some of the problems this approach might face. The translation of the co-reflection rule seems to be quite challenging, one might approach this by formalizing it as $A_a(P) \rightarrow \mathsf{K}_a(P)$, read as if an agent is aware of a proof of P, she knows that P. However it is not clear, how semantics for this operator should work.

It might be easier to obtain a completeness result using a different notion of model. For example Coquand and Smith (1996) proves completeness of topological models for intuitionistic FOL (first-order-logic) in a proof assistant based on constructive type theory. In unpublished work, Krupski also mentioned that IEL has the finite model property for topological models.

In the context of cut-elimination proofs it might be interesting to try and find even easier proofs or to prove cut for a general class of logics. For example Tews (2013) formalizes cut-elimination for the class co-algebraic logics in Coq, based on a paper by Pattinson and Schröder (2010). However classical axioms are used.

There is the option to prove decidability by embedding (intuitionistic) propositional modal logics into the two-variable guarded fragment of first-order logic (Alechina & Shkatov, 2006). Formalizing such a translation and the decidability of the two-variable guarded fragment would immediately yield decidability proofs for a large class of modal logics, however this is non-trivial.

8.3 *Overview of the Mechanization*

The mechanization also includes some results which have not been further mentioned in the thesis. These are the aforementioned soundess proof of Tarau's embedding and a embedding of IEL into Coq's logic which can be used to establish consistency of a variant of the natural deduction system in $\mathbb{T}$ instead of $\mathbb{P}$. We believe that it should in theory be possible to translate between the two for consistency proofs, since the goal is negated. The complete development has been verified using Coq 8.13.2. The development and a reference to the accompanying documentation can be found online, the accompanying coq-development can also be found there.

Component	Spec	Proof
preliminaries	121	93
natural deduction + lindenbaum	183	418
completeness	219	585
constructive completeness	81	258
cut-elimination + decidability IEL	193	398
cut-elimination + decidability K	116	362
$\sum$	720	2307
permutation-based cut for K	125	644
permutation-based cut for IEL	176	1045
permutation library and solver	106	143
$\sum$	407	1832
Overall $\sum$	1127	4139

Figure 8.3.1: Overview of the mechanization components

In Figure 8.3.1 an overview over the size of the development in terms of code size is given. The line-counts were extracted using coqwc. Reiterating a point made in the discussion at the end of chapter 6, the difference in code size between the permutation encoding and the permutation-encoding by Dang and Smolka is vast. While our cut-elimination proofs differ significantly from Dang's, the decidability proofs are similar but had to be slightly adjusted for the modal rules and different formula datatype.

Our formulization also relied on a permutation-solver from GitHub[1] and the Programming System Lab's Base Library[2]. It is an interesting prospective for future work to optimize the compile-time by using automatic proof search / solvers more efficiently. While the current compile time is feasible, it could be faster. On a 2.6 GHz machine compilation takes about 4 minutes, however when compiling with 4 threads the compilation time reduces to only 2.5 minutes. This is mostly due to proof automation taking a long time to solve for example (more or less easy) inclusions e.g. $\Gamma_1, A \subseteq \Gamma_1, \Gamma_3$ if we additionally know that $A \in \Gamma_1$. The hope would be that for example proofs of the inversion lemmas could be fully automated. Here it might help to use more advanced Coq features, for example, creating multiple hint databases.

[1] https://github.com/foreverbell/permutation-solver

[2] https://github.com/uds-psl/base-library

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