Memo: Cut Elimination for IEL in Coq

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We present a proof of cut elimination for IEL. The proof is an adaption of the proof given in Krupski and Yatmanov (2016) and has been mechanized in the Coq proof assistant. The basic proof structure, i.e. an induction on pairs of natural numbers to proof that any cut can directly be eliminated remains is inspired by Plato (2001).

In Krupski and Yatmanov (2016) decidability and cut-elimination for IEL are proven. The presented proof is similar to the proof of cut-elimination for IPC given in Troelstra and Schwichtenberg (2000). Formalizing this proof in a proof assistant involves design decisions about representing the contexts (multisets of formulae in the paper) and reasoning about them. While these technical details are interesting, we feel it is better to give the proof without elaborating too many of the somewhat gory details about the formalization. We proceed in the following fashion: We present the proof on the same abstraction level as textbooks about structural proof theory would do, only sometimes hinting at parts which are more difficult to prove using a proof assistant. After presenting the proof, we investigate the representation of the sequent calculus in Coq and the collection of facts about multisets which was needed in the development.

Cut Elimination: History

The first cut elimination proof was given by Gentzen (1935a, 1935b). TBD.

Existing formalizations of Cut Elimination Formalizing cut elimination proofs for a sequent calculus is not a new topic. Michaelis and Nipkow (2017) formalize a cut elimination proof in Isabelle/HOL closely following Troelstra and Schwichtenberg (2000) (but using induction on derivations instead of induction of height multiple times). Penington (2018) tried to do a similiar proof following Troelstra & Schwichtenberg in Coq, but ultimately failed, relying on a structural multiset [1] encoding. A structural encoding is also used in Park (2013) master thesis, where cut elimination and the termination of a bidirectional proof search for an intuitionistic modal logic are proven. However with this encoding the proofs are quite complicated and generate a fairly big coq development with many case distinctions on list equivalences. van Doorn (2015) formalized a semantic cut elimination proof for classical propositional logic in Coq using this encoding. The idea to use multisets explicitely for a cut elimination proof is used in Chaudhuri, Lima, and Reis (2017), where cut elimination for an intuitionistic logic in Abella is proven.

[1] (c.f. Section Formalization)

For IPC simpler formalized cut elimination proofs exist, where the cut elimination proof is a simple structural induction using a generalized cut elimination theorem (like Gentzen MultiCut) by Smolka and Brown (2012) for the implicational fragment of IPC and later by

Dang (2015) for full IPC. It relies on using the sequent calculus G3Imp described in Troelstra and Schwichtenberg (2000) which is very well suited for proof search (TODO: Explanation why it is so well suited).

Sequent Calculus for IEL

The sequent calculus for IEL is derived from the calculus G3ip as presented in Troelstra and Schwichtenberg (2000). Of course new rules for the **K** operator are introduced. Krupski and Yatmanov (2016) distinguish multiple equivalent systems with different rules, however all of these are equivalent and only ease the proof of equivalence betweem the natural deduction system and the sequent calculus. We use their sequent calculus IELGM (and the version with depts IELGMd).

Only two rules are added: The KI-rule, which absorbs both the intuitionistic reflection ($A \to \mathbf{K}A$) and K-distributivity rule from the natural deduction calculus for IEL. And the $\mathbf{K}\bot$-rule, which basically represents the truth condition in IEL.

In fig. 1 a presentation of the rules is given.

Figure 1: IELGM sequent calculus, the naming conventions match the coq formalization, i.e. the left rule for $\wedge$ is called AL for and-left instead of $\wedge$L.

$$\frac{\bot \in \Gamma}{\Gamma \Rightarrow s} \qquad \frac{p_i \in \Gamma}{\Gamma \Rightarrow p_i} \qquad \frac{A, B, \Gamma \Rightarrow s}{A \wedge B, \Gamma \Rightarrow s}$$

$$\frac{\Gamma \Rightarrow s \qquad \Gamma \Rightarrow t}{\Gamma \Rightarrow s \wedge t}\ (\mathrm{AR}) \qquad \frac{s, \Gamma \Rightarrow u \qquad t, \Gamma \Rightarrow t}{(s \wedge t), \Gamma \Rightarrow u}\ (\mathrm{AL})$$

$$\frac{\Gamma \Rightarrow F_i}{\Gamma \Rightarrow F_1 \vee F_2}\ (\mathrm{OR}_i) \qquad \frac{S, \Gamma \Rightarrow F \qquad T, \Gamma \Rightarrow F}{S \vee T, \Gamma \Rightarrow F}\ (\mathrm{OL})$$

$$\frac{F, \Gamma \Rightarrow G}{\Gamma \Rightarrow F_1 \supset F_2}\ (\mathrm{IR}) \qquad \frac{S \supset T, \Gamma \Rightarrow S \qquad T, \Gamma \Rightarrow F}{S \supset T, \Gamma \Rightarrow F}\ (\mathrm{IL})$$

$$\frac{\mathbf{K}(\Delta), \Delta, \Gamma \Rightarrow \phi}{\Gamma, \mathbf{K}(\Delta) \Rightarrow \mathbf{K}\phi}\ (\mathrm{KI}) \qquad \frac{\Gamma \Rightarrow \mathbf{K}\bot}{\Gamma \Rightarrow F}\ (\mathrm{KB})$$

For a derivation of a formula F from a multiset of premisses Γ we use the common notation $\Gamma \Rightarrow F$.

With $\overset{n}{\Rightarrow}$ we denote, that a derivation of height less or equal to n exists. The height of a derivation is the maximum of the subderivation's heights increased by 1. We encode the height in the inductive rules in the same way as done in Michaelis and Nipkow (2017). That is we explicitly have a step rule which allows to arbitrarily increase the height of every derivation by one (and of course by iterated application by any natural number). This additional rule allows us to formalize every other rule with premises of the same height. For example we could define the AR-Rule with heights explicitely as

$$\frac{\Gamma \overset{n}{\Rightarrow} s \qquad \Gamma \overset{n}{\Rightarrow} t}{\Gamma \overset{n+1}{\Rightarrow} s \wedge t}\ \text{ARd}$$

, combining this scheme with the step rule has the same effect as using the maximum over the subderivation's heights in the constructor [2] e.g.

[2] However reasoning with maximum of heights is needed when proving that from any IELGM derivation a IELGMd derivation can be obtained.

$$\frac{\Gamma \overset{n}{\Rightarrow} s \qquad \Gamma \overset{m}{\Rightarrow} t}{\Gamma \overset{\max(n,m)+1}{\Rightarrow} s \wedge t}\ \text{ARd'}$$

.

Lemma 1. *If* $\Gamma \Rightarrow F$ *iff there exists an* n *s.t.* $\Gamma \overset{n}{\Rightarrow} F$.

Lemma 2 (Weakening). *IELGM enjoys the depth preserving weakening property:* $\Gamma \overset{n}{\Rightarrow} F \implies G, \Gamma \overset{n}{\Rightarrow} F$.

Proof. Structural induction on the derivation (just as Michaelis and Nipkow (2017); Troelstra and Schwichtenberg (2000) use induction on n). □

We can prove some inversion rules, used for proving contraction.

Lemma 3. *Left rules are invertible in the following sense:*

- $A \wedge B, \Gamma \Rightarrow C \implies A, B, \Gamma \Rightarrow C$.
- $A \vee B, \Gamma \Rightarrow C \implies A, \Gamma \Rightarrow C$ *and* $B, \Gamma \Rightarrow C$
- $A \supset B, \Gamma \Rightarrow C \implies B, \Gamma \Rightarrow C$

Proof. The proofs are by structural induction on n , as in Troelstra and Schwichtenberg (2000). □

The last result we need is the contraction proof. Contraction allows us to ignore duplicate assumptions. We will proof it using the standard form.

Lemma 4 (Contraction). *If* $F, F, \Gamma \overset{n}{\Rightarrow} s$ *then* $F, \Gamma \overset{n}{\Rightarrow} s$.

Proof. We do an induction on n with s and Γ quantified. □

For completeness sake, we state the contraction lemma in the way we will use it: Multisets can be collapsed into sets. For example if we have a proof that $\Gamma_1, \Gamma_1, \Gamma_2 \Rightarrow s$, we also have a proof that $\Gamma_1, \Gamma_2 \Rightarrow s$.

Lemma 5 (Useful contraction). *If* $A \subseteq B$ *and* $B \subseteq A$, $A \Rightarrow s \iff B \Rightarrow s$.[3]

[3] Inclusion here is to be read as set-inclusion, not multiset-inclusion.

Proof. With informal reasoning, it is a simple consequence of above lemma. When reasoning in a proof assistant, a reduction to duplicate-free lists is needed. □

With these lemmas inplace, we can prove the main result.

Lemma 6 (Cut is admissible). *If* $\Gamma_1 \Rightarrow F$ *and* $F, \Gamma_2 \Rightarrow \Delta$ *then* $\Gamma_1, \Gamma_2 \Rightarrow \Delta$.

Proof. We do an induction on pairs (r, k) of natural numbers where r is the cut-rank of the cut-derivation, that is the sum of the premisses heights and where k is the length of the formula [4], such that we obtain two inductive hypotheses: Any cut with lower cut-rank or same cut-rank and cutformula of a lower size can be eliminated by the inductive hypothesis. We do not do an induction on the derivation, but instead only do an inversion i.e. a case analysis on which derivation was used to derive the left premiss. This suffices for most cases, in some cases we also need a second case distinction on the derivation of the right premiss afterwards.

[4] The length of a formula is defined inductively.

$$\begin{aligned} l(p_i) &= 0 \\ l(\bot) &= 0 \\ l(s \circ t) &= 1 + \max(l(s), l(t)) \\ l(\mathbf{K}s) &= 1 + l(s) \end{aligned}$$

, where $\circ$ is any binary operator. Note, that subformulas are always of a smaller size.

We will proof some of the cases in detail here.

1. The left premiss was derived using the $\bot$-Rule, thus our derivation has the following form [5] :

$$\dfrac{\dfrac{\bot \in \Gamma_1}{\Gamma_1 \Rightarrow F} \qquad F, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \Gamma_2 \Rightarrow \Delta}$$

   Since $\bot \in \Gamma_1$, we know that there is a context Γ_1' s.t. $\Gamma_1 \equiv \bot, \Gamma_1'$. Therefore we can use weakening and the bot introduction rule.

[5] We omit the heights of the derivations, since they are not needed in this case.

2. If the left premiss was derived using the variable-rule (therefore $F \in \Gamma_1$), we can use weakening and the right derivation.

3. Assume the left premiss was derived using the left rule for $\wedge$, thus there exists Γ_1 s.t. $A_1 \wedge A_2, \Gamma_1' = \Gamma_1$.

$$\dfrac{\dfrac{A_1, A_2, \Gamma_1' \overset{n-1}{\Rightarrow} F}{A_1 \wedge A_2, \Gamma_1' \overset{n}{\Rightarrow} F} \qquad \Gamma_2 \overset{m}{\Rightarrow} \Delta}{\Gamma_1, \Gamma_2 \Rightarrow \Delta}$$

   We can do a cut of lower height on the same formula F by permuting the application of the left introduction rule for $\wedge$ down.

$$\dfrac{\dfrac{A_1, A_2, \Gamma1' \overset{n-1}{\Rightarrow} F \qquad \Gamma_2 \overset{n}{\Rightarrow} \Delta}{A_1, A_2, \Gamma_1' \overset{n}{\Rightarrow} \Delta}\text{ r-cut}}{A_1 \wedge A_2, \Gamma_1', \Gamma_2 \overset{m}{\Rightarrow} \Delta}$$

   Since $(n-1) + n < n + m$, we can apply the IH and remove the cut.

4. Assume the left premiss was derive using the right rule for $\wedge$:

$$\dfrac{\dfrac{\Gamma1 \overset{n-1}{\Rightarrow} A_1 \qquad \Gamma1 \overset{n-1}{\Rightarrow} A_2}{\Gamma_1 \overset{n}{\Rightarrow} A_1 \wedge A_2} \qquad A_1 \wedge A_2, \Gamma_2 \overset{n}{\Rightarrow} \Delta}{\Gamma_1, \Gamma_2 \overset{S(n)}{\Rightarrow} \Delta}$$

   By the inversion lemma for $\wedge$, we can obtain a derivation $A_1, A_2, \Gamma_2 \overset{n}{\Rightarrow} \Delta$. We will replace the cut with 2 cuts with a formula of lower size (we cut on A_1 and A_2 instead of $A_1 \wedge A_2$). Our new derivation has the following form:

$$\dfrac{\Gamma_1 \overset{n-1}{\Rightarrow} A_1 \qquad \dfrac{\Gamma_1 \overset{n-1}{\Rightarrow} A_2 \qquad F, G, \Gamma_2 \overset{n-1}{\Rightarrow} \Delta}{A_1, \Gamma_1, \Gamma_2 \overset{n}{\Rightarrow} \Delta}}{\Gamma_1, (\Gamma_1, \Gamma_2) \overset{S(n)}{\Rightarrow} \Delta}$$

5. Assume the left premiss was derived using the left for disjunction.

$$\dfrac{\dfrac{A_1, \Gamma_1' \Rightarrow F \qquad A_2, \Gamma_1' \Rightarrow F}{A_1 \lor A_2, \Gamma 1' \overset{n-1}{\Rightarrow} A_2} \qquad F, \Gamma_2 \overset{n}{\Rightarrow} \Delta}{\Gamma_1, \Gamma_2 \overset{S(n)}{\Rightarrow} \Delta}$$

   We can permute the cut upwards.

$$\dfrac{\dfrac{A_1, \Gamma_1' \Rightarrow F \qquad F, \Gamma_2 \Rightarrow \Delta}{A_1, \Gamma_1', \Gamma_2 \Rightarrow \Delta} \qquad \dfrac{A_2, \Gamma_1' \Rightarrow F \qquad F, \Gamma_2 \Rightarrow \Delta}{A_2, \Gamma_1', \Gamma_2 \Rightarrow F}}{A_1 \lor A_2, \Gamma_1', \Gamma_2 \Rightarrow \Delta}$$

   Note, that both cuts have a lower cutrank since their left premiss has a derivation which is 1 less deep.

6. Assume the left premiss was derived using one of the right-rules for disjunction. We can cut on the smaller formula.

7. Assume the premiss was derived using the left introduction rule for implication. Permute the cut upwards, just as in the disjunction case.

8. Assume the premiss was derived using the right introduction rule for implication. We need to do a second case analysis on the derivation $F, \Gamma_2 \Rightarrow \Delta$. (We only prove select cases here)

   (a) If the second premiss is an axiom, either $F = \Delta$ or $\Delta \in \Gamma_2$. In both cases the cut is unnecessary.

   (b) Similarly, if the second premiss is derived using the bottom rule, either $F = \bot$ or $\bot \in \Delta$.

   (c) An interesting case arises, when the right premiss is proved using the left introduction rule for implication.

$$\dfrac{\dfrac{s_0, \Gamma_1 \Rightarrow t_0}{\Gamma_1 \Rightarrow s_0 \supset t_0} \qquad \dfrac{\overset{[D_2]}{s_1 \supset t_1, s_0 \supset t_0, \Gamma_2' \Rightarrow s_1} \qquad t_1, s_0 \supset t_0, \Gamma_2' \Rightarrow \Delta}{s_1 \supset t_1, s_0 \supset t_0, \Gamma_2' \Rightarrow \Delta}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta}$$

   We have 2 cases: Either $s_0 \supset t_0 = s_1 \supset t_1$ or $s_0 \supset t_0 \neq s_1 \supset t_1$ and .

   i. In the first case we can build the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma_1 \Rightarrow s_0 \supset t_0 \qquad s_0 \supset t_0, \Gamma_2 \Rightarrow s_1}{\Gamma_1, \Gamma_2 \Rightarrow s_1}\,\text{r-cut} \qquad \dfrac{s_1, \Gamma_1 \Rightarrow t_1 \qquad t_1, \Gamma_2 \Rightarrow \Delta}{(s_1, \Gamma_1), \Gamma_2 \Rightarrow \Delta}\,\text{s-cut}}{(\Gamma_1, \Gamma_2), (\Gamma_1, \Gamma_2) \Rightarrow \Delta}\,\text{s-cut}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta}\,\text{contr.}$$

ii. If $s_0 \supset t_0 \neq s_1 \supset t_1$ a multiset Γ_2' exists s.t. $\Gamma_2 = (s_1 \supset t_1), \Gamma_2'$.

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow s_0 \supset t_0 \qquad s_0 \supset t_0, \Gamma_2 \Rightarrow s_1}{\Gamma_1, (s_1 \supset t_1, \Gamma_2') \Rightarrow s_1}\text{s-cut} \qquad \dfrac{\Gamma_1 \Rightarrow s_0 \supset t_0 \qquad s_0 \supset t_0, t_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, t_1, \Gamma_2' \Rightarrow \Delta}\text{r-cut}}{\dfrac{s_1 \supset t_1, \Gamma_1, \Gamma_2' \Rightarrow \Delta}{\Gamma_1, \Gamma_2 \Rightarrow \Delta}\text{rewriting}}\text{IL}$$

9. Assume the premiss was derived using the K-introduction rule. We do a second case distinction on the derivation of the right deduction. Most cases are similiar to those obtained in the right rule for implication subcases, we won't go into too much detail here.

(a) The right premise is an axiom. Either $\Delta = F$, in which case weakening and using IH on the left deduction suffices or $\Delta \in \Gamma_2$ in which case we can directly construct the derivation.

(b) The most interesting case occurs, when the KI-rule is used on both sides . To ease the notation we change the notation (Note: I guess it might be better to do this globally): We rename Δ into Ω.

So assume $F = \mathbf{K}F', \Omega = \mathbf{K}\,\Omega'$ and $\Gamma_1 = \Gamma_1', \mathbf{K}(\Delta_1)$ and $F, \Gamma_2 = \Gamma_2', \mathbf{K}(\Delta_2)$. We have the following derivation:

$$\dfrac{\dfrac{\Gamma_1', \Delta_1, \mathbf{K}(\Delta_1) \Rightarrow F'}{\Gamma_1', \mathbf{K}(\Delta_1) \Rightarrow \mathbf{K}\,F'} \qquad \dfrac{\Gamma_2, \Delta_2, \mathbf{K}(\Delta_2) \Rightarrow \Omega'}{\Gamma_2, \mathbf{K}(\Delta_2) \Rightarrow \mathbf{K}\Omega'}}{\Gamma_1, \Gamma_2 \Rightarrow \mathbf{K}\Omega'}$$

We do a case analysis on $F \in \Gamma_2' \vee F \in \mathbf{K}(\Delta_2)$.

i. Assume $F \in \Gamma_2'$. Therefore there is a Γ_2'' s.t. $\Gamma_2 = \Gamma_2'', \mathbf{K}(\Delta_2)$.

$$\dfrac{\dfrac{\dfrac{\Gamma_1, \Delta_1, \mathbf{K}(\Delta_1) \Rightarrow F \qquad F, \Gamma_2'', \Delta_2 \mathbf{K}(\Delta_2) \Rightarrow \Omega'}{\Gamma_1, \Delta_1, \mathbf{K}(\Delta_1), \Gamma_2', \mathbf{K}(\Delta_2) \Rightarrow \Omega}\text{r-cut}}{\Gamma_1', \Gamma_2'', \mathbf{K}(\Delta_1, \Delta_2) \Rightarrow \mathbf{K}\Omega'}\text{KI}}{\Gamma_1, \Gamma_2 \Rightarrow \mathbf{K}\Omega'}\text{rewriting}$$

We first apply the K-introduction rule and afterwards cut on F.

ii. Assume $F \in \mathbf{K}(\Delta_2)$. Therefore $F' \in \Delta_2$ and there is a Δ_2' s.t. $\Delta_2 = F', \Delta_2'$. [6].

$$\dfrac{\dfrac{\dfrac{\Gamma_1, \Delta_1, \mathbf{K}(\Delta_1) \Rightarrow F' \qquad \dfrac{\Gamma_1, \mathbf{K}(\Delta_1) \Rightarrow \mathbf{K}\,F \qquad \Gamma_2, \Delta_2, \mathbf{K}(\Delta_2) \Rightarrow \Omega'}{\Delta_2', F', \mathbf{K}(\Delta_2) \Rightarrow \Omega'}\text{r-cut}}{\Gamma_1', \Gamma_2', \Delta_1, \Delta_2', \mathbf{K}(\Delta_1), \mathbf{K}(\Delta_2') \Rightarrow \Omega'}\text{s-cut}}{\Gamma_1', \Gamma_2', \mathbf{K}(\Delta_1, \Delta_2') \Rightarrow \mathbf{K}\Omega'}}{\Gamma_1, \Gamma_2 \Rightarrow \mathbf{K}\Omega'}\text{rewriting}$$

[6] This reasoning is suprisingly complicated in a proof assistant (injectivity + map needed), -> formalization section

10. Assume the premiss was derived using the K-bottom rule.

$$\dfrac{\dfrac{\Gamma_1 \overset{n-1}{\Rightarrow} \mathbf{K}\bot}{\Gamma_1 \overset{n}{\Rightarrow} F} \qquad F, \Gamma_2 \overset{n}{\Rightarrow} \Delta}{\Gamma_1, \Gamma_2 \Rightarrow \Delta}$$

We use weakening and the KB rule.

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow \mathbf{K}\bot}{\Gamma_1, \Gamma_2 \Rightarrow K\bot}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta}$$

□

Equivalence to nd

We first present the proof that any nd-derivation can be turned into a sequent calculus derivation. We first proof some left elimination results for the sequent calculus. The proofs are easy inductions.

Lemma 7. $s, \Gamma \Rightarrow t \iff \Gamma \Rightarrow s \supset t$

Proof. The only-if direction is a trivial consequence of the right introduction rule for $\supset$. We proof the left direction by induction on the derivation with Γ quantified. □

A inversion rule can be shown for and.

Lemma 8. $\Gamma \Rightarrow s \land t \iff \Gamma \Rightarrow s \land \Gamma \Rightarrow t$

Proof. Induction on the derivation. □

We can also show, that the deduction system has the reflexivity property.

Lemma 9 (Reflexivity for $\Rightarrow$)**.** $s, \Gamma \Rightarrow s$

Proof. The proof is by induction on the formula s with Γ quantified.

1. If s is $\bot$ or a variable, it is trivial.
2. Assume $s = a \land b$. We can build the following derivation:
3. The other cases, are similiar, we will take a look at the $s = \mathbf{K}s'$ case.

□

With these lemmas in-place we can proof the result.

Lemma 10 (Sc $\implies$ Nd)**.** *If* $\Gamma \vdash s$*, then* $\Gamma \Rightarrow s$

Proof. Induction on the derivation, we only cover some select cases.

- If the assumption rule was used, we can apply reflexivity (lemma 9).
- Assume the $A \vdash$ **Ks** was derived from $A \vdash s$. By the inductive hypothesis. we obtain a sc-derivation $A \Rightarrow s$. Since $A \equiv A + +[]$, we can apply the K-Introduction rule.
- Assume the k-implication rule was used i.e. we get an inductive hypothesis $\Gamma \Rightarrow \mathbf{K}(s \rightarrow t)$ and have to proof $\Gamma \Rightarrow \mathbf{K}s \rightarrow \mathbf{K}t$. We first apply the cut-rule with $\mathbf{K}(s \supset t)$. Therefore we need to complete a derivation of $K(s \supset t), \Gamma \Rightarrow \mathbf{K}s \supset \mathbf{K}t$. Using the right-rule for implication, we are left with proving $\mathbf{K}(s \supset t), \mathbf{K}s\Gamma \Rightarrow \mathbf{K}s \supset \mathbf{K}t$. We can now use the KI-rule and only need to prove $s, s \supset t, \mathbf{K}s, \mathbf{K}(s \supset t), \Gamma \Rightarrow t$, which is simple (using cut and reflexivity).

All other cases are easily solved by using the elimination lemmas, just as in Troelstra and Schwichtenberg (2000). □

Since we have proven both directions of the equivalence, we get the desired result, that the sequent calculus formulation of IEL really is equivalent to the natural deduction system.

Formalization

Multisets

We represent multisets as lists. Using lists as the representation for sets or multisets of formulas is common, however there seem to be atleast two possibilities for how to deal with list equivalence.

1. Adopt the rules s.t. they need not refer to a special list equality (we call this a structural encoding [7]).
2. Embed list equality into the rules.

[7] Is there a better term for this?

To analyze the difference between the two approaches, let us consider the and-elimination rule. If encoded in a structural way, we would state in the following form: For any list L_1, L_2 over formulas if $L_1 + +F \wedge G + +L2 \Rightarrow s$, then $L_1 + +F :: G :: L2 \Rightarrow s$. That is by squeezing the formula between arbitrary lists of formulas we embed multiset equivalence implicitely.

This approach is not feasible for IEL, since it is impossible to embed the K-Introduction rule in this fashion (since it is not just about a single element somewhere in the multiset, but the whole mutliset being equivalent to a multiset union). Aditionally with the structural representations some proofs get more complex [8]. The alternative we are using is to define multiset equivalence explicetely and use in the rules. This makes the proofs more similiar to textbook proofs, but has the drawback that a solid collection of multiset lemmata is needed (going beyond the lemmata contained in the standard library). For example case analysis on multiset equivalences is needed, as are some registering multisets for rewriting in Coq.

[8] tries to prove cut elimination for IPC using this approach but does not succeed,in cut elimination is proven, however not via a syntactic approach.

In this section we will discuss some of the additional theorems needed. We make use of the uds-psl base library for duplicate free lists and some facts about inclusions.

Decidability

TODO: Literatureinordnung

A fixpoint theorem

Our proof search will rely on a fixpoint theorem. We introduce some terminology, let $f : \mathbb{N} \to A \to \mathbb{B}$ be a function, where A is an arbitrary type.

We call f monotone with a respect to a subset $U \subseteq A$ if and only if for every $n \in \mathbb{N}$ and every $u \in U$, $f(n) \leq f(n+1)$.[9] We say

[9] We assume that $\perp \leq \top$.

that f makes progress at n in $\mathcal{U}$ if and only if there is an $u \in \mathcal{U}$ s.t. $f(n, u) \neq f(n+1, u)$. Note that if f is monotone, it can only make progress if at least one element *changes* from true to false.

With these terms we can now define the crucial property for our proof search. f has the progress-property if f only makes progress at $n+1$ if it made progress at n (with respect to $\mathcal{U}$.

Definition 1. *f has the noprogress-property with respect to $U \subseteq A$ iff $\forall n, noprogress(f, U, n) \to noprogress(f, U, n+1)$.*

Intuitively the noprogress property guarentess, that the values of the function won't change, if they have not changed in one step.

References

Chaudhuri, K., Lima, L., & Reis, G. (2017). Formalized Meta-Theory of Sequent Calculi for Substructural Logics. *Electronic Notes in Theoretical Computer Science*, *332*, 57–73. Retrieved from `http://dx.doi.org/10.1016/j.entcs.2017.04.005` doi: 10.1016/j.entcs.2017.04.005

Dang, H. (2015). *Systems for Propositional Logics* (Tech. Rep.). Saarland University. Retrieved from `https://www.ps.uni-saarland.de/{~}dang/ri-lab/propsystems/systems.pdf`

Gentzen, G. (1935a). Untersuchungen über das logische Schließen. I. *Mathematische Zeitschrift*. doi: 10.1007/BF01201353

Gentzen, G. (1935b). Untersuchungen über das logische Schließen. II. *Mathematische Zeitschrift*. doi: 10.1007/BF01201363

Krupski, V. N., & Yatmanov, A. (2016). Sequent calculus for intuitionistic epistemic logic IEL. In *Lecture notes in computer science (including subseries lecture notes in artificial intelligence and lecture notes in bioinformatics)* (Vol. 9537, pp. 187–201). Springer Verlag. Retrieved from `https://link.springer.com/chapter/10.1007/978-3-319-27683-0{_}14` doi: 10.1007/978-3-319-27683-0_14

Michaelis, J., & Nipkow, T. (2017). Formalized Proof Systems for Propositional Logic. (5), 1–5.

Park, H. (2013). *Bidirectional Proof Search Procedure for Intuitionistic Modal Logic IS5* (Bachelors Thesis, Pohang University of Science and Technology (POSTECH)). Retrieved from `http://pl.postech.ac.kr/IS5/thesis1.pdf`

Penington, J. (2018). *Classical Propositional Proof Theory Proving in Coq Joseph Penington* (Tech. Rep. No. May).

Plato, J. V. (2001). A proof of Gentzen ' s Hauptsatz without multicut. , *24*, 9–18.

Smolka, G., & Brown, C. E. (2012). *Introduction to Computational Logic*. Retrieved from `http://www.ps.uni-saarland.de/courses/cl-ss12/script/icl.pdf`

Troelstra, A. S., & Schwichtenberg, H. (2000). *Basic Proof Theory*. doi: 10.1017/cbo9781139168717 van Doorn, F. (2015). Propositional Calculus in Coq. , 1–11. Retrieved from `http://arxiv.org/abs/1503.08744`