Assignment 2
Semantics, WS 2013/14

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Definitions: $I := \lambda x.x$, $B := \lambda fgx.f(gx)$, $\hat{n} := \lambda fx.f^n x$, $\text{succ} := \lambda nfx.f(nfx)$.

Exercise 2.1 Prove the following equivalences by hand:

a) $\hat{n} \equiv \lambda fx.\hat{n}\ f\ x$

b) $\hat{n} \equiv \lambda f.\hat{n}\ f$

c) $\text{succ}\ s\ t \equiv B\ t\ (s\ t)$

d) $\widehat{Sn} \equiv \text{succ}\ \hat{n}$

e) $\widehat{Sn}\ s\ t \equiv s\ (\hat{n}\ s\ t)$

f) $\widehat{Sn}\ s \equiv B\ s\ (\hat{n}\ s)$

g) $\hat{n}\ s \equiv \hat{n}\ (B\ s)\ I$

h) $\widehat{m+n} \equiv \lambda s.B\ (\widehat{m}\ s)\ (\hat{n}\ s)$

i) $\widehat{m \cdot n} \equiv B\ \widehat{m}\ \hat{n}$

j) $\widehat{m^n} \equiv \hat{n}\ (B\ \widehat{m})\ \hat{1}$

Exercise 2.2 Define the following functions on Church numerals. You may use primitive recursion (prec).

a) Tetration: ${}^n a = \underbrace{a^{a^{\cdot^{\cdot^{a}}}}}_{n}$. We leave the value of ${}^0 a$ unspecified.

b) Ackermann's function:

$$A(x,y) = \begin{cases} y+1 & \text{if } x = 0 \\ A(x-1,1) & \text{if } y = 0 \\ A(x-1, A(x, y-1)) & \text{otherwise} \end{cases}$$

If you run into difficulties, read Chapter 4 in the ICL lecture notes.

Exercise 2.3 Consider the following two definitions of multiplication.

$$\text{add} := \lambda m\,n\,f\,x.mf(nfx)$$
$$\text{mul}_1 := \lambda m\,n\,f.m(nf)$$
$$\text{mul}_2 := \lambda m\,n.m(\text{add}\ n)(\lambda fx.x)$$

The definitions agree on Church numerals, but are not equivalent. Find terms $a_1, \dots, a_k$ such that:

$$\text{mul}_1\, a_1 \dots a_k \succ^* \lambda xy.x$$
$$\text{mul}_2\, a_1 \dots a_k \succ^* \lambda xy.y$$

Exercise 2.4 Let s be an abstraction such that x is not free in s. Argue that $\lambda x.sx \equiv s$.

Exercise 2.5 (Coq) Show that the Church-Rosser Property (CRP) implies uniqueness of normal forms.