Assignment 5
Semantics, WS 2013/14

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Read in the lecture notes: Chapter 2.1 - 2.7 and step through the Coq development ARS.v

Exercise 5.1 Carry out the power definition of R^* in Coq and prove the equivalence with the linear definition.

$$\begin{aligned} R^* &:= \bigcup_{n\in\mathbb{N}} R^n \\ R^0 &:= \{\,(x,x) \mid x \in X\,\} \\ R^{n+1} &:= R \circ R^n \\ R \circ S &:= \{\,(x,z) \mid \exists y.\ Rxy \land Syz\,\} \end{aligned}$$

Exercise 5.2 Prove the following properties of $R^{\equiv}$ in Coq.

a) Monotonicity: $R \preccurlyeq S \to R^{\equiv} \preccurlyeq S^{\equiv}$

b) Minimality: If $R \preccurlyeq S$ and S is an equivalence, then $R^{\equiv} \preccurlyeq S$.

c) Idempotence: $(R^{\equiv})^{\equiv} \approx R^{\equiv}$

Exercise 5.3 Prove that diamond, confluence and Church-Rosser are extensional properties.

Define a non-extensional predicate *sym* such that

$$Ext_2 \to (sym\ R \leftrightarrow symmetric\ R)$$

where $Ext_2 = \forall RS, R \approx S \to R = S$.

Exercise 5.4 Prove that every semi-confluent relation is Church-Rosser. Start with a diagram-based proof sketch, give the textual proof, and finally do the proof with Coq.

Exercise 5.5

a) Let ρ be an idempotent and monotone function mapping relations into relations. Show that $R \preccurlyeq S \preccurlyeq \rho R \to \rho R \approx \rho S$.

b) Show $R \preccurlyeq S \preccurlyeq R^* \to R^* \approx S^*$

c) Show $R \preccurlyeq S \preccurlyeq R^{\equiv} \to R^{\equiv} \approx S^{\equiv}$

Exercise 5.6 Two relations R, S commute (*com* R S) if for $R\ x\ y$ and $S\ x\ z$ there exists u such that $S\ y\ u$ and $R\ z\ u$. We have *diamond* R = *com* R R and *confluent* R = *diamond* (R^*) = *com* (R^*) (R^*).

a) Show *com* R S → *com* S (R^*).

b) Use (a) to conclude that *com* R S → *com* (R^*) (R^*).

c) Using Exercise 5.5 and part (b), show the commutative union lemma: Given two confluent, commuting relations R and S, their union $R \cup S$ is confluent.

Exercise 5.7 Find a locally confluent relation that is not confluent.

Exercise 5.8 Establish the canonical induction principle for R^* in Coq.

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Lemma star_canonical_ind R (p : X -> Prop) y :
  p y ->
  (forall x x', R x x' -> star R x' y -> p x' -> p x) ->
  forall x, star R x y -> p x
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