Assignment 7
Semantics, WS 2013/14

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Read in the lecture notes: Chapter 2 & 3

Repertoire

Exercise 7.1 (IP.v) Study the intersection model for inductive predicates at the example of the evenness predicate using Coq.

a) Define predicates *D1*, *D2*, and *DI* such that

$$spec\ q \ := \ D1\ q \land D2\ q \land DI\ q$$

is a specification for evenness predicates.

b) Show that the specification *spec* has at most one solution up to equivalence.

c) Define *DL* and prove

$$DI\ p \to DL\ p$$
$$D1\ p \to D2\ p \to DL\ p \to DI\ p$$

d) Define the predicate

$$even\ n \ := \ \forall p.\ D1\ p \to D2\ p \to p\,n$$

and show that it satisfies the specification *spec*.

e) Prove the following facts.

(i) $even\ 4$

(ii) $\neg even\ 1$

(iii) $even\ (S(S\ n)) \to even\ n$

(iv) $even\ n \to \neg even\ (Sn)$

Hint: Use the tactic *refine* to apply the induction principle. Note that (ii) and (iv) can be shown with the induction principle *L*, while (iii) requires the induction principle *BI*.

f) Define an evenness predicate using an inductive definition and prove that it satisfies the specification *spec*.

Exercise 7.2 (IP.v) Let a type X, a predicate $R : X \to X \to Prop$ and a point $a : X$ be given. We define a predicate "R can reach a from x" inductively:

$$\frac{}{reach\ R\ a\ a} \qquad\qquad \frac{Rxy \qquad reach\ R\ a\ y}{reach\ R\ a\ x}$$

a) Define the predicate *reach R a* with the intersection method in Coq and show that it satisfies the base lemmas coming with the inference rules.

b) Define R^* with a native inductive definition in Coq and prove $R^* \approx reach\ R$.

Exercise 7.3 (IP.v) Let a type X and a predicate $R : X \to X \to Prop$ be given. We define a predicate "R terminates on x" inductively:

$$\frac{Rx \preceq ter\ R}{ter\ R\ x}$$

a) Define the predicate $ter\ R$ with the intersection method in Coq and show that it satisfies the base lemmas coming with the inference rule.
b) Define $SN\ R$ with a native inductive definition in Coq and prove $SN\ R \approx ter\ R$.

Exercise 7.4 (CL.v) Study the definition of term, step, redex, termi, pstep and ρ and prove the following:

a) $\forall s, termi\ s$
b) $dec(reducible\ step)$
c) $reflexive\ pstep$
d) $step \le pstep$
e) $s \succeq^* s' \to t \succeq^* t' \to st \succeq^* s't'$
f) $pstep \le step^*$
g) $pstep^* \approx step^*$
h) $triangle\ pstep\ \rho \to church\_rosser\ step$
i) $triangle\ pstep\ \rho$

Extra

Exercise 7.5 (IP.v) Let X be a type and $F : (X \to Prop) \to (X \to Prop)$ be a monotone predicate (i.e., $\forall p\ q.\ p \preceq q \to Fp \preceq Fq$). Find a predicate I such that you can prove the following. The intersection $p \sqcap q$ abbreviates the predicate $\lambda x.\ px \wedge qx$.

a) $Fp \preceq p \to I \preceq p$
b) $FI \preceq I$
c) $FI \approx I$
d) $p \preceq I \to Fp \preceq I$
e) $F(I \sqcap p) \preceq I$
f) $F(I \sqcap p) \preceq p \leftrightarrow I \preceq p$

Hint: The problem is a translation of a special case of the Knaster-Tarski fixed point theorem from set theory to type theory. Google to find out more about the theorem and its proof. The proof of the Knaster-Tarski theorem is a classical example for the use of the intersection method in Mathematics.

Challenge

Exercise 7.6 Let R, S be SN with $R \cup S$ transitive. Show that $R \cup S$ is SN. You may use classical logic ($\forall P, P \vee \neg P$).