On the expressive power of effect handlers and monadic reflection

Yannick Forster
Master's thesis supvervised by Ohad Kammar and Marcelo Fiore

How to incorporate effects?

- global store (i.e. references),
- exceptions,
- I/O,
- random,
- nondeterminism,
- or concurrency

An example

```
exception Error
val r = ref 0
fun error () = raise Error

fun test () = (r := 5; error() handle Error => !r)
```

`test()` evaluates to?

An example

```
exception Error
val r = ref 0
fun error () = raise Error

fun test () = (r := 5; error() handle Error => !r)
```

`test()` evaluates to?

Why not to `0`?

WOULD BE COOL:

User definable effects on top of a functional language

WOULD BE COOL:

User definable effects on top of a functional language

- Monads
- Algebraic effects
- Delimited control

Goal

Compare two existing approaches in their expressiveness
Fine-grained notion needed, because most practical languages are Turing-complete

Goal

Compare two existing approaches in their expressiveness
Fine-grained notion needed, because most practical languages are Turing-complete

More like
“Compare expressiveness of recursion and while-loops”
than like
“Compare expressiveness of Coq and Javascript”

Approach

- take a base language (functional, typed, strongly normalising)
- add each concept to the language
- define denotational semantics to each resulting calculus
- prove denotational semantics to be adequate
- use this to compare expressiveness

Take a base language

Call-by-push-value lambda-calculus from Levy

Distinguishes between values and computations

Levy (1999), Levy (2004)

Take a base language

$$
\begin{array}{rrcl}
\text{(values)} & V, W & ::= & x \mid () \mid (V_1, V_2) \mid \mathbf{inj}_i\, V \mid \{M\} \\
\text{(computations)} & & & \\
& M, N & ::= & \mathbf{split}(V, x_1.x_2.M) \mid \mathbf{case_0}(V) \\
& & \mid & \mathbf{case}(V, x_1.M_1, x_2.M_2) \mid V! \\
& & \mid & \mathbf{return}\ V \mid \mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N \\
& & \mid & \lambda x.M \mid M\ V \\
& & \mid & \langle M_1, M_2 \rangle \mid \mathbf{prj}_i\, M
\end{array}
$$

Figure 2.1: CBPV syntax

Distinguishes between values and computations

Levy (1999), Levy (2004)

Take a base language

$$
\begin{array}{rrcl}
\text{(values)} & V, W & ::= & x \mid () \mid (V_1, V_2) \mid \mathbf{inj}_i\, V \mid \{M\} \\
\text{(computations)} & & & \\
& M, N & ::= & \mathbf{split}(V, x_1.x_2.M) \mid \mathbf{case_0}(V) \\
& & \mid & \mathbf{case}(V, x_1.M_1, x_2.M_2) \mid V! \\
& & \mid & \mathbf{return}\ V \mid \mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N \\
& & \mid & \lambda x.M \mid M\ V \\
& & \mid & \langle M_1, M_2 \rangle \mid \mathbf{prj}_i\, M
\end{array}
$$

Figure 2.2: CBPV syntax

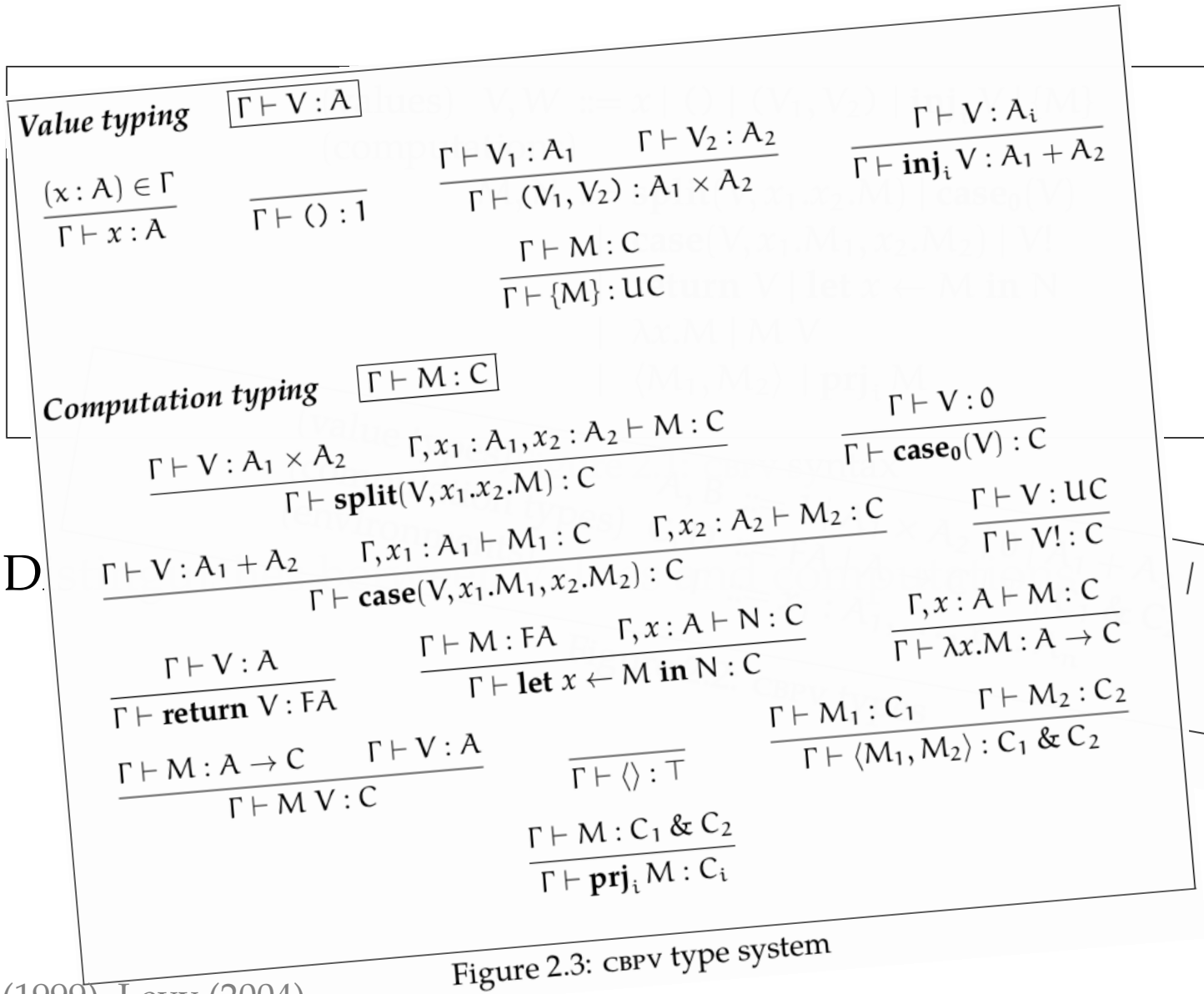
Distinguish between values and computations

$$
\begin{array}{rrcl}
\text{(value types)} & A, B & ::= & 1 \mid A_1 \times A_2 \mid 0 \mid A_1 + A_2 \mid UC \\
\text{(computation types)} & C, D & ::= & FA \mid A \to C \mid \top \mid C_1 \,\&\, C_2 \\
\text{(environments)} & \Gamma & ::= & x_1 : A_1, \ldots, x_n : A_n
\end{array}
$$

Figure 2.2: CBPV types

Levy (1999), Levy (2004)

Take a base language



Value typing $\boxed{\Gamma \vdash V : A}$

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{}{\Gamma \vdash () : 1} \qquad \frac{\Gamma \vdash V_1 : A_1 \qquad \Gamma \vdash V_2 : A_2}{\Gamma \vdash (V_1, V_2) : A_1 \times A_2} \qquad \frac{\Gamma \vdash V : A_i}{\Gamma \vdash \mathbf{inj}_i\, V : A_1 + A_2}$$

$$\frac{\Gamma \vdash M : C}{\Gamma \vdash \{M\} : UC}$$

Computation typing $\boxed{\Gamma \vdash M : C}$

$$\frac{\Gamma \vdash V : A_1 \times A_2 \qquad \Gamma, x_1 : A_1, x_2 : A_2 \vdash M : C}{\Gamma \vdash \mathbf{split}(V, x_1.x_2.M) : C} \qquad \frac{\Gamma \vdash V : 0}{\Gamma \vdash \mathbf{case}_0(V) : C}$$

$$\frac{\Gamma \vdash V : A_1 + A_2 \qquad \Gamma, x_1 : A_1 \vdash M_1 : C \qquad \Gamma, x_2 : A_2 \vdash M_2 : C}{\Gamma \vdash \mathbf{case}(V, x_1.M_1, x_2.M_2) : C} \qquad \frac{\Gamma \vdash V : UC}{\Gamma \vdash V! : C}$$

$$\frac{\Gamma \vdash V : A}{\Gamma \vdash \mathbf{return}\, V : FA} \qquad \frac{\Gamma \vdash M : FA \qquad \Gamma, x : A \vdash N : C}{\Gamma \vdash \mathbf{let}\, x \leftarrow M\, \mathbf{in}\, N : C} \qquad \frac{\Gamma, x : A \vdash M : C}{\Gamma \vdash \lambda x.M : A \to C}$$

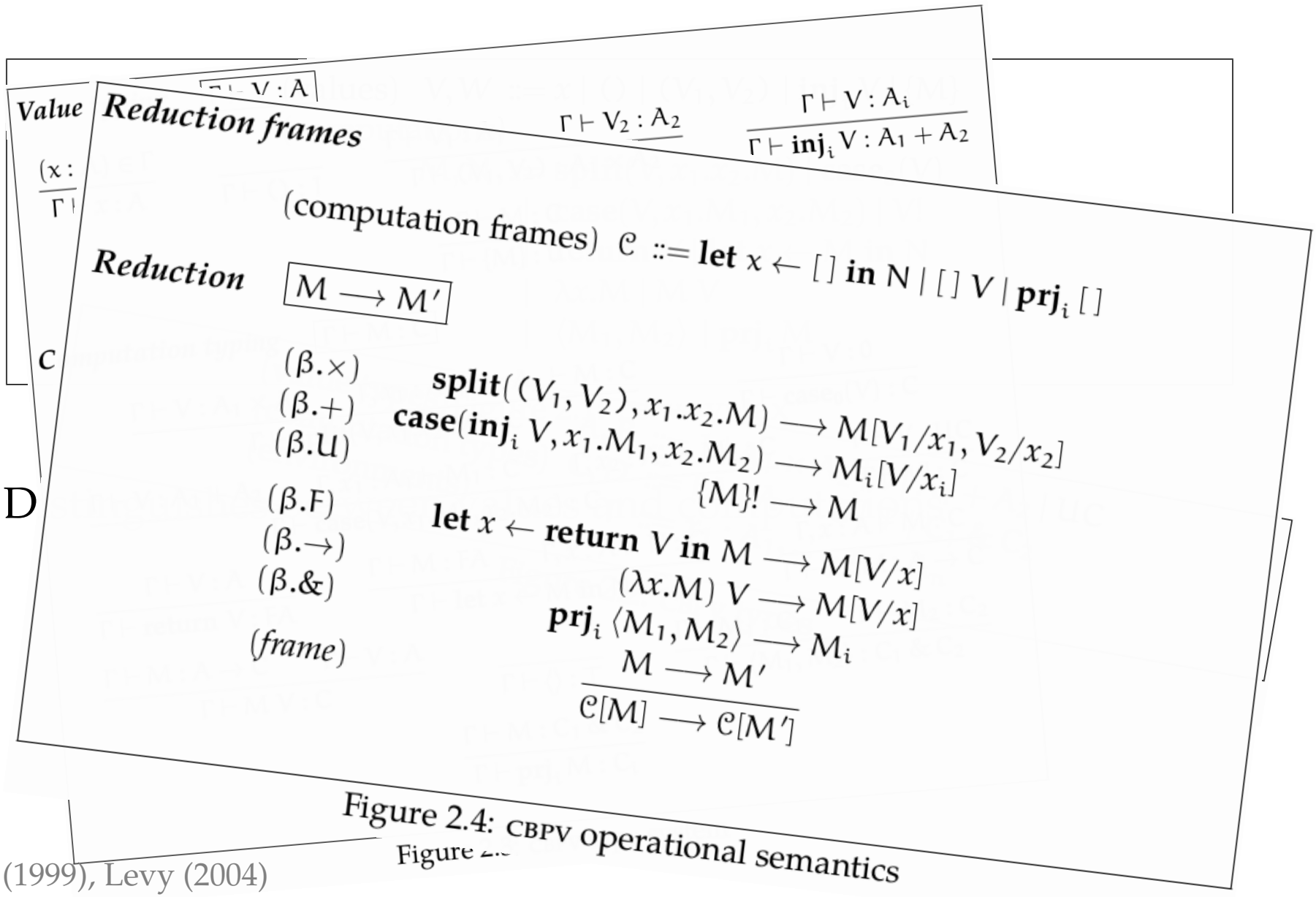
$$\frac{\Gamma \vdash M : A \to C \qquad \Gamma \vdash V : A}{\Gamma \vdash M\, V : C} \qquad \frac{}{\Gamma \vdash \langle\rangle : \top} \qquad \frac{\Gamma \vdash M_1 : C_1 \qquad \Gamma \vdash M_2 : C_2}{\Gamma \vdash \langle M_1, M_2 \rangle : C_1 \,\&\, C_2}$$

$$\frac{\Gamma \vdash M : C_1 \,\&\, C_2}{\Gamma \vdash \mathbf{prj}_i\, M : C_i}$$

Figure 2.3: CBPV type system

Levy (1999), Levy (2004)

Take a base language



Value

$$\frac{\Gamma \vdash V : A_1 \qquad \Gamma \vdash V_2 : A_2}{\Gamma \vdash (V_1, V_2) : A_1 \times A_2} \qquad \frac{\Gamma \vdash V : A_i}{\Gamma \vdash \mathbf{inj}_i\, V : A_1 + A_2}$$

Reduction frames

(computation frames) $\mathcal{C} ::= \mathbf{let}\ x \leftarrow [\,]\ \mathbf{in}\ N \mid [\,]\ V \mid \mathbf{prj}_i\ [\,]$

Reduction $M \longrightarrow M'$

$$(\beta.\times)\quad \mathbf{split}((V_1, V_2), x_1.x_2.M) \longrightarrow M[V_1/x_1, V_2/x_2]$$
$$(\beta.+)\quad \mathbf{case}(\mathbf{inj}_i\ V, x_1.M_1, x_2.M_2) \longrightarrow M_i[V/x_i]$$
$$(\beta.U)\quad \{M\}! \longrightarrow M$$
$$(\beta.F)\quad \mathbf{let}\ x \leftarrow \mathbf{return}\ V\ \mathbf{in}\ M \longrightarrow M[V/x]$$
$$(\beta.\to)\quad (\lambda x.M)\ V \longrightarrow M[V/x]$$
$$(\beta.\&)\quad \mathbf{prj}_i\ \langle M_1, M_2\rangle \longrightarrow M_i$$
$$(\mathit{frame})\quad \frac{M \longrightarrow M'}{\mathcal{C}[M] \longrightarrow \mathcal{C}[M']}$$

Figure 2.4: CBPV operational semantics

Figure 2.

Levy (1999), Levy (2004)

ADD EACH CONCEPT

- Effects and handler calculus λ_{eff}
- Monadic reflection calculus λ_{mon}

Kammar, Lindley, and Oury (2013), Filinski (2010)

Effect handlers

Exception handlers:

```
(... (raise e n : B) ...) : C
  handle e (n : A) => (exp : C)
```

Effect handlers

Exception handlers:

```
(... (raise e n : B) ...) : C
   handle e (n : A) => (exp : C)
```

Effect handlers:

```
(... (e n : B) ...) : A
   handle e (name : string, k : B -> A) =>
   (exp2 : A)
```

The calculus λ_{eff}

$$
\begin{array}{lll}
\text{(values)} & V, W & ::= x \mid () \mid (V_1, V_2) \mid \mathbf{inj}_i\, V \mid \{M\} \\
\text{(computations)} & & \\
& M, N & ::= \mathbf{split}(V, x_1.x_2.M) \mid \mathbf{case}_0(V) \\
& & \mid\ \mathbf{case}(V, x_1.M_1, x_2.M_2) \mid V! \\
& & \mid\ \mathbf{return}\ V \mid \mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N \\
& & \mid\ \lambda x.M \mid M\ V \\
& & \mid\ \langle M_1, M_2 \rangle \mid \mathbf{prj}_i\, M \\
& & \mid\ \mathrm{op}\ V(\lambda x.M) \mid \mathbf{handle}\ M\ \mathbf{with}\ H \\
\text{(handlers)} & H & ::= \{\mathbf{return}\ x \mapsto M\} \\
& & \mid\ H \uplus \{\mathrm{op}\ p\ k \mapsto N\} \quad \text{where op does not occur in } H
\end{array}
$$

Figure 3.1: λ_{eff}-calculus syntax

The calculus λ_{eff}

$$V, W ::= x \mid () \mid (V_1, V_2) \mid \mathbf{inj}_i\, V \mid \{M\}$$

$$\mathbf{split}(V, x_1.x_2.M) \mid \mathbf{case}_0(V)$$

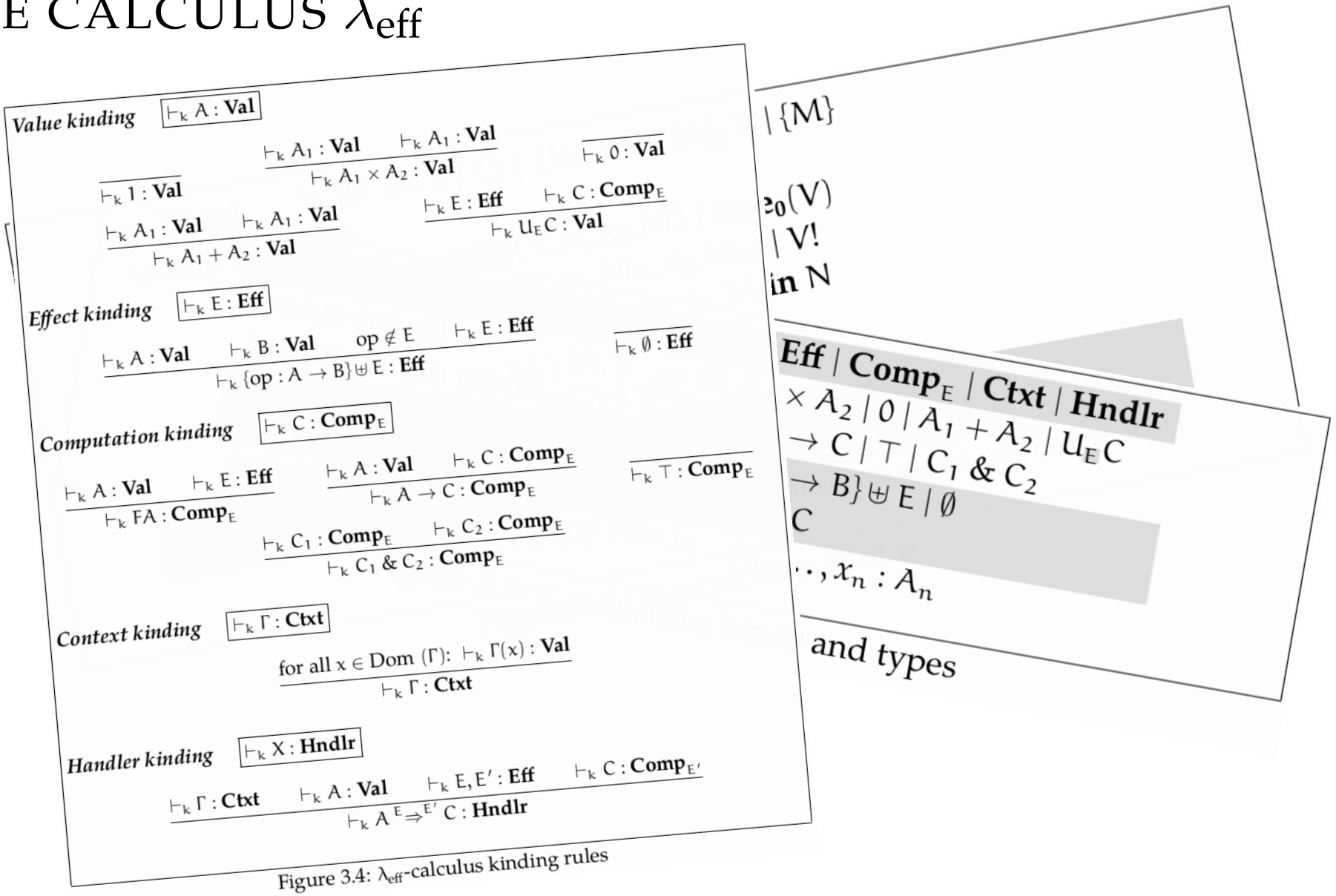
$$M_1, x_2.M_2) \mid V!$$

$$\_ \leftarrow M \text{ in } N$$

(kinds)	$K ::= \mathbf{Val} \mid \mathbf{Eff} \mid \mathbf{Comp}_E \mid \mathbf{Ctxt} \mid \mathbf{Hndlr}$
(value types)	$A, B ::= 1 \mid A_1 \times A_2 \mid 0 \mid A_1 + A_2 \mid U_E\, C$
(computation types)	$C, D ::= FA \mid A \to C \mid \top \mid C_1 \,\&\, C_2$
(effect signatures)	$E ::= \{\mathrm{op} : A \to B\} \uplus E \mid \emptyset$
(handler types)	$R ::= A \,{}^{E}\!\Rightarrow^{E'} C$
(environments)	$\Gamma ::= x_1 : A_1, \ldots, x_n : A_n$

Figure 3.3: λ_{eff}-calculus kinds and types

The calculus λ_{eff}



Value kinding $\boxed{\vdash_k A : \mathbf{Val}}$

$$\frac{}{\vdash_k 1 : \mathbf{Val}} \qquad \frac{\vdash_k A_1 : \mathbf{Val} \quad \vdash_k A_1 : \mathbf{Val}}{\vdash_k A_1 \times A_2 : \mathbf{Val}} \qquad \frac{}{\vdash_k 0 : \mathbf{Val}}$$

$$\frac{\vdash_k A_1 : \mathbf{Val} \quad \vdash_k A_1 : \mathbf{Val}}{\vdash_k A_1 + A_2 : \mathbf{Val}} \qquad \frac{\vdash_k E : \mathbf{Eff} \quad \vdash_k C : \mathbf{Comp}_E}{\vdash_k U_E C : \mathbf{Val}}$$

Effect kinding $\boxed{\vdash_k E : \mathbf{Eff}}$

$$\frac{\vdash_k A : \mathbf{Val} \quad \vdash_k B : \mathbf{Val} \quad \mathrm{op} \notin E \quad \vdash_k E : \mathbf{Eff}}{\vdash_k \{\mathrm{op} : A \to B\} \uplus E : \mathbf{Eff}} \qquad \frac{}{\vdash_k \emptyset : \mathbf{Eff}}$$

Computation kinding $\boxed{\vdash_k C : \mathbf{Comp}_E}$

$$\frac{\vdash_k A : \mathbf{Val} \quad \vdash_k E : \mathbf{Eff}}{\vdash_k FA : \mathbf{Comp}_E} \qquad \frac{\vdash_k A : \mathbf{Val} \quad \vdash_k C : \mathbf{Comp}_E}{\vdash_k A \to C : \mathbf{Comp}_E} \qquad \frac{}{\vdash_k \top : \mathbf{Comp}_E}$$

$$\frac{\vdash_k C_1 : \mathbf{Comp}_E \quad \vdash_k C_2 : \mathbf{Comp}_E}{\vdash_k C_1 \,\&\, C_2 : \mathbf{Comp}_E}$$

Context kinding $\boxed{\vdash_k \Gamma : \mathbf{Ctxt}}$

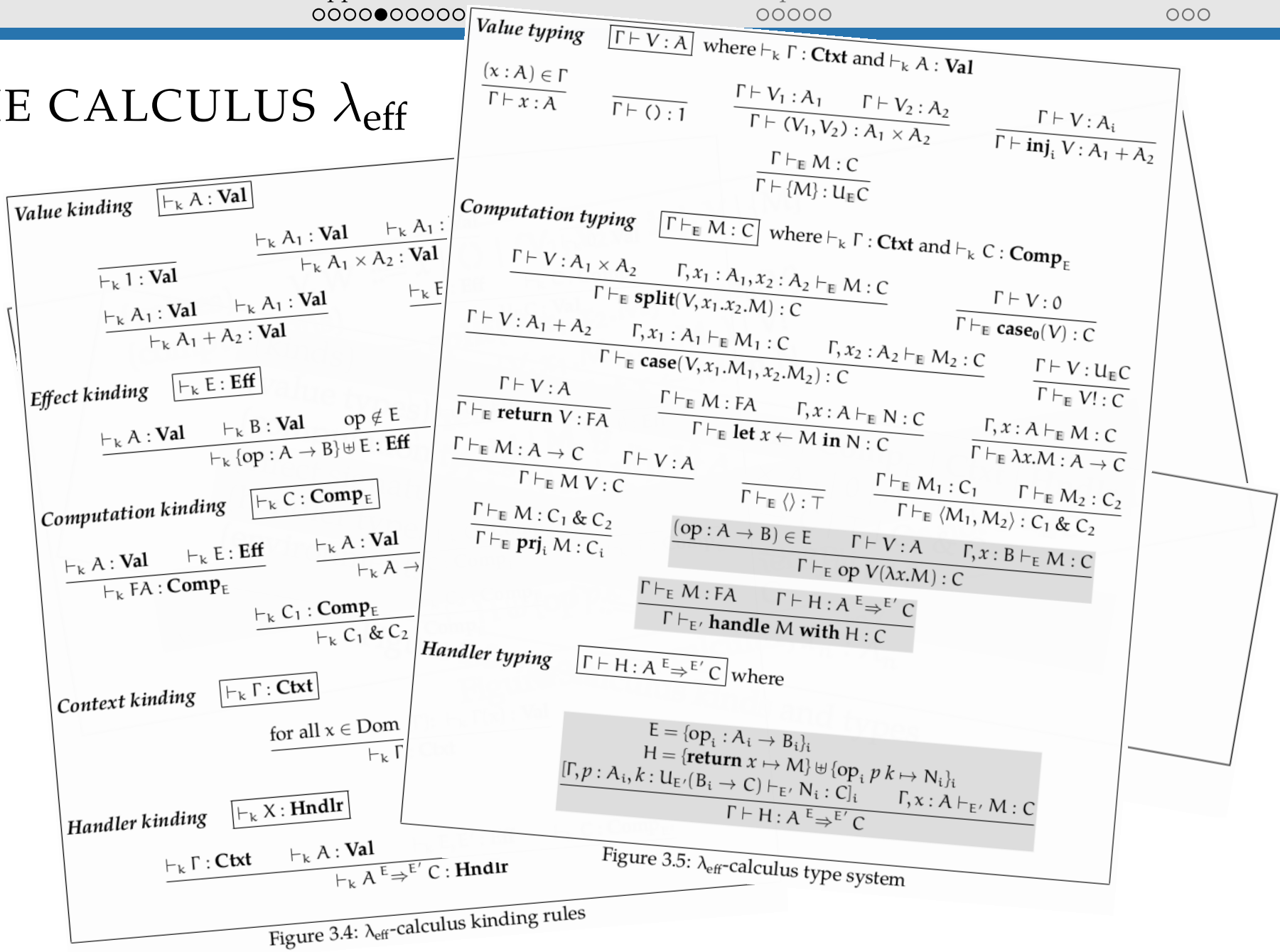
$$\frac{\text{for all } x \in \mathrm{Dom}\,(\Gamma)\text{: } \vdash_k \Gamma(x) : \mathbf{Val}}{\vdash_k \Gamma : \mathbf{Ctxt}}$$

Handler kinding $\boxed{\vdash_k X : \mathbf{Hndlr}}$

$$\frac{\vdash_k \Gamma : \mathbf{Ctxt} \quad \vdash_k A : \mathbf{Val} \quad \vdash_k E, E' : \mathbf{Eff} \quad \vdash_k C : \mathbf{Comp}_{E'}}{\vdash_k A\ ^{E}\!\Rightarrow^{E'} C : \mathbf{Hndlr}}$$

Figure 3.4: λ_{eff}-calculus kinding rules

The calculus λ_{eff}



Value kinding $\boxed{\vdash_k A : \mathbf{Val}}$

$$\frac{}{\vdash_k 1 : \mathbf{Val}} \qquad \frac{\vdash_k A_1 : \mathbf{Val} \qquad \vdash_k A_1 :}{\vdash_k A_1 \times A_2 : \mathbf{Val}}$$

$$\frac{\vdash_k A_1 : \mathbf{Val} \qquad \vdash_k A_1 : \mathbf{Val}}{\vdash_k A_1 + A_2 : \mathbf{Val}} \qquad \vdash_k E$$

Effect kinding $\boxed{\vdash_k E : \mathbf{Eff}}$

$$\frac{\vdash_k A : \mathbf{Val} \qquad \vdash_k B : \mathbf{Val} \qquad \mathrm{op} \notin E}{\vdash_k \{\mathrm{op} : A \to B\} \uplus E : \mathbf{Eff}}$$

Computation kinding $\boxed{\vdash_k C : \mathbf{Comp}_E}$

$$\frac{\vdash_k A : \mathbf{Val} \qquad \vdash_k E : \mathbf{Eff}}{\vdash_k FA : \mathbf{Comp}_E} \qquad \frac{\vdash_k A : \mathbf{Val}}{\vdash_k A \to}$$

$$\frac{\vdash_k C_1 : \mathbf{Comp}_E}{\vdash_k C_1 \,\&\, C_2}$$

Context kinding $\boxed{\vdash_k \Gamma : \mathbf{Ctxt}}$

$$\frac{\text{for all } x \in \mathrm{Dom}}{\vdash_k \Gamma}$$

Handler kinding $\boxed{\vdash_k X : \mathbf{Hndlr}}$

$$\frac{\vdash_k \Gamma : \mathbf{Ctxt} \qquad \vdash_k A : \mathbf{Val}}{\vdash_k A \,{}^{E}\!\Rightarrow^{E'} C : \mathbf{Hndlr}}$$

Figure 3.4: λ_{eff}-calculus kinding rules

Value typing $\boxed{\Gamma \vdash V : A}$ where $\vdash_k \Gamma : \mathbf{Ctxt}$ and $\vdash_k A : \mathbf{Val}$

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{}{\Gamma \vdash () : 1} \qquad \frac{\Gamma \vdash V_1 : A_1 \qquad \Gamma \vdash V_2 : A_2}{\Gamma \vdash (V_1, V_2) : A_1 \times A_2} \qquad \frac{\Gamma \vdash V : A_i}{\Gamma \vdash \mathbf{inj}_i\, V : A_1 + A_2}$$

$$\frac{\Gamma \vdash_E M : C}{\Gamma \vdash \{M\} : U_E C}$$

Computation typing $\boxed{\Gamma \vdash_E M : C}$ where $\vdash_k \Gamma : \mathbf{Ctxt}$ and $\vdash_k C : \mathbf{Comp}_E$

$$\frac{\Gamma \vdash V : A_1 \times A_2 \qquad \Gamma, x_1 : A_1, x_2 : A_2 \vdash_E M : C}{\Gamma \vdash_E \mathbf{split}(V, x_1.x_2.M) : C} \qquad \frac{\Gamma \vdash V : 0}{\Gamma \vdash_E \mathbf{case}_0(V) : C}$$

$$\frac{\Gamma \vdash V : A_1 + A_2 \qquad \Gamma, x_1 : A_1 \vdash_E M_1 : C \qquad \Gamma, x_2 : A_2 \vdash_E M_2 : C}{\Gamma \vdash_E \mathbf{case}(V, x_1.M_1, x_2.M_2) : C} \qquad \frac{\Gamma \vdash V : U_E C}{\Gamma \vdash_E V! : C}$$

$$\frac{\Gamma \vdash V : A}{\Gamma \vdash_E \mathbf{return}\ V : FA} \qquad \frac{\Gamma \vdash_E M : FA \qquad \Gamma, x : A \vdash_E N : C}{\Gamma \vdash_E \mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N : C} \qquad \frac{\Gamma, x : A \vdash_E M : C}{\Gamma \vdash_E \lambda x.M : A \to C}$$

$$\frac{\Gamma \vdash_E M : A \to C \qquad \Gamma \vdash V : A}{\Gamma \vdash_E M\, V : C} \qquad \frac{}{\Gamma \vdash_E \langle\rangle : \top} \qquad \frac{\Gamma \vdash_E M_1 : C_1 \qquad \Gamma \vdash_E M_2 : C_2}{\Gamma \vdash_E \langle M_1, M_2 \rangle : C_1 \,\&\, C_2}$$

$$\frac{\Gamma \vdash_E M : C_1 \,\&\, C_2}{\Gamma \vdash_E \mathbf{prj}_i\, M : C_i} \qquad \frac{(\mathrm{op} : A \to B) \in E \qquad \Gamma \vdash V : A \qquad \Gamma, x : B \vdash_E M : C}{\Gamma \vdash_E \mathrm{op}\, V(\lambda x.M) : C}$$

$$\frac{\Gamma \vdash_E M : FA \qquad \Gamma \vdash H : A \,{}^{E}\!\Rightarrow^{E'} C}{\Gamma \vdash_{E'} \mathbf{handle}\ M\ \mathbf{with}\ H : C}$$

Handler typing $\boxed{\Gamma \vdash H : A \,{}^{E}\!\Rightarrow^{E'} C}$ where

$$\frac{\begin{array}{c} E = \{\mathrm{op}_i : A_i \to B_i\}_i \\ H = \{\mathbf{return}\ x \mapsto M\} \uplus \{\mathrm{op}_i\, p\, k \mapsto N_i\}_i \\ [\Gamma, p : A_i, k : U_{E'}(B_i \to C) \vdash_{E'} N_i : C]_i \end{array} \qquad \Gamma, x : A \vdash_{E'} M : C}{\Gamma \vdash H : A \,{}^{E}\!\Rightarrow^{E'} C}$$

Figure 3.5: λ_{eff}-calculus type system

The calculus λ_{eff}

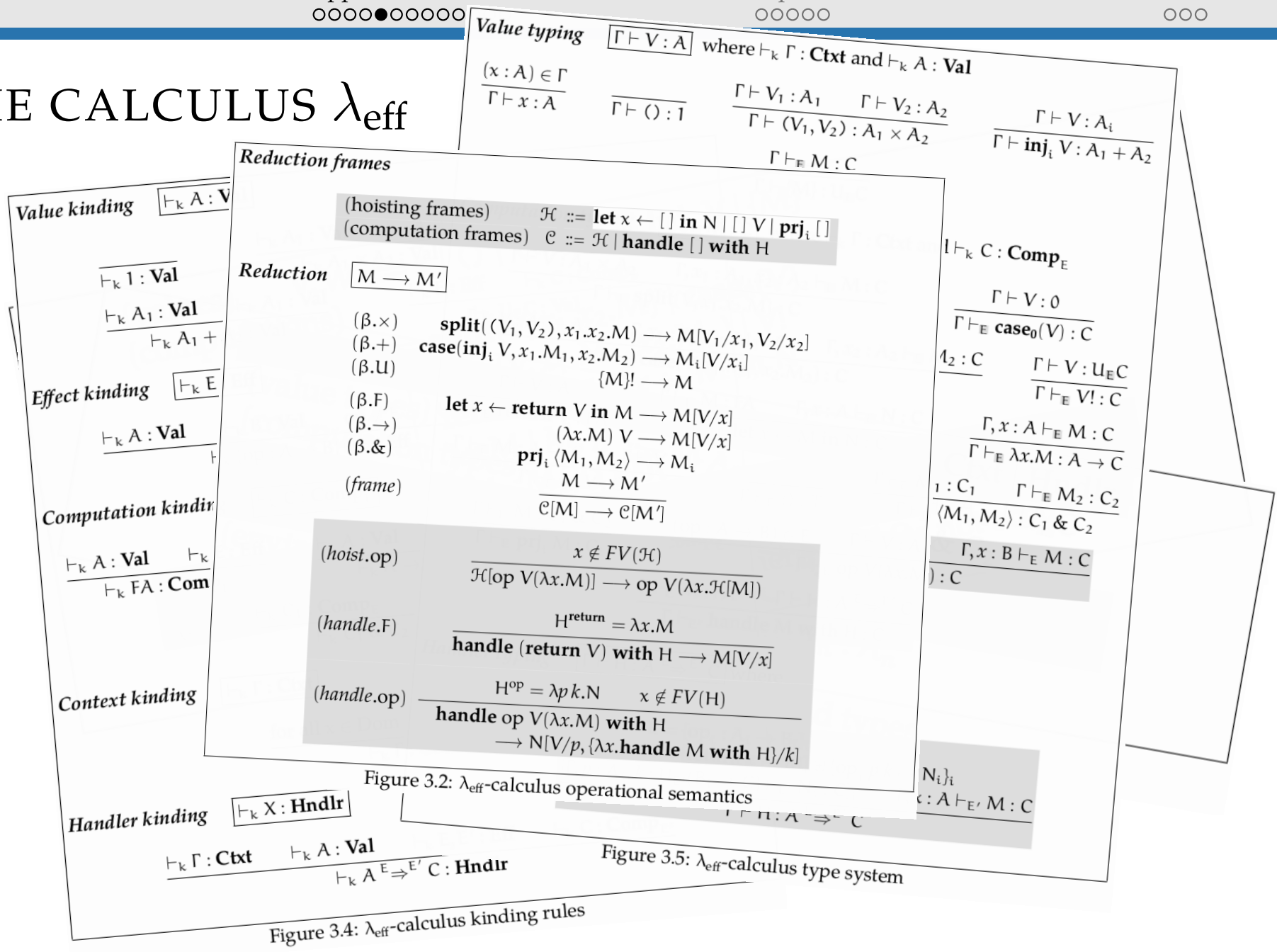


Figure 3.2: λ_{eff}-calculus operational semantics

Figure 3.4: λ_{eff}-calculus kinding rules

Figure 3.5: λ_{eff}-calculus type system

Programming with effect handlers

$$\texttt{raise} : \text{string} \to FA$$

$$\texttt{try} : U_{\{\text{exc:string}\to 0\}} FA \to U_E(\text{string} \to FA) \to FA$$

Programming with effect handlers

$$\texttt{raise} : \text{string} \to FA$$

$$\texttt{try} : U_{\{\text{exc:string}\to 0\}} FA \to U_E(\text{string} \to FA) \to FA$$

$$\texttt{try}\{1 + \texttt{raise}\ \text{"number"}\}\{\lambda s.\textbf{if}\ s = \text{"number"}\ \textbf{then}\ 0\ \textbf{else}\ 1\}$$

Programming with effect handlers

$$\texttt{raise} : \text{string} \to FA$$

$$\texttt{try} : U_{\{\text{exc:string}\to 0\}} FA \to U_E(\text{string} \to FA) \to FA$$

$$\texttt{try}\{1 + \texttt{raise}\ \text{"number"}\}\{\lambda s.\textbf{if}\ s = \text{"number"}\ \textbf{then}\ 0\ \textbf{else}\ 1\}$$

$$\longrightarrow^* 0$$

Programming with effect handlers

$$\texttt{raise} \coloneqq \lambda s.\mathbf{let}\ x \leftarrow \mathrm{exc}\ s\ (\lambda x.\mathbf{return}\ x)\ \mathbf{in}\ \mathbf{case_0}(x)$$

$$\texttt{try} \coloneqq \lambda c\ h.\mathbf{handle}\ c!\ \mathbf{with}\ H$$

where $H^{\mathbf{return}} \coloneqq \lambda x.\mathbf{return}\ x$ and $H^{\mathrm{exc}} \coloneqq \lambda s\ k.h!\ s.$

Monadic reflection

Error monad: $TA := A + 1$

Monadic reflection

Error monad: $TA := A + 1$
Reflecion for computations with errors:

$$\frac{M : 1 \xrightarrow{\text{err}} A}{\texttt{reify}_{\text{err}}(M) : A + 1} \qquad \frac{N : A + 1}{\texttt{reflect}_{\text{err}}(N) : 1 \xrightarrow{\text{err}} A}$$

Effects are ordered with the trivial effect $\bot$.

The calculus λ_{mon}

$$
\begin{array}{rcl}
\text{(values)} \quad V, W & ::= & x \mid () \mid (V_1, V_2) \mid \mathbf{inj}_i\, V \mid \{M\} \\
\text{(computations)} & & \\
M, N & ::= & \mathbf{split}(V, x_1.x_2.M) \mid \mathbf{case}_0(V) \\
& \mid & \mathbf{case}(V, x_1.M_1, x_2.M_2) \mid V! \\
& \mid & \mathbf{return}\ V \mid \mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N \\
& \mid & \lambda x.M \mid M\ V \\
& \mid & \langle M_1, M_2 \rangle \mid \mathbf{prj}_i\, M \\
& \mid & [N]^{\varepsilon} \mid \widehat{\mu}^{\varepsilon}(N) \\
& \mid & \mathbf{leteffect}\ \varepsilon \succ e\ \mathbf{be}\ (\alpha.C, N_u, N_b)\ \mathbf{in}\ N
\end{array}
$$

Figure 4.1: λ_{mon}-calculus syntax

The calculus λ_{mon}

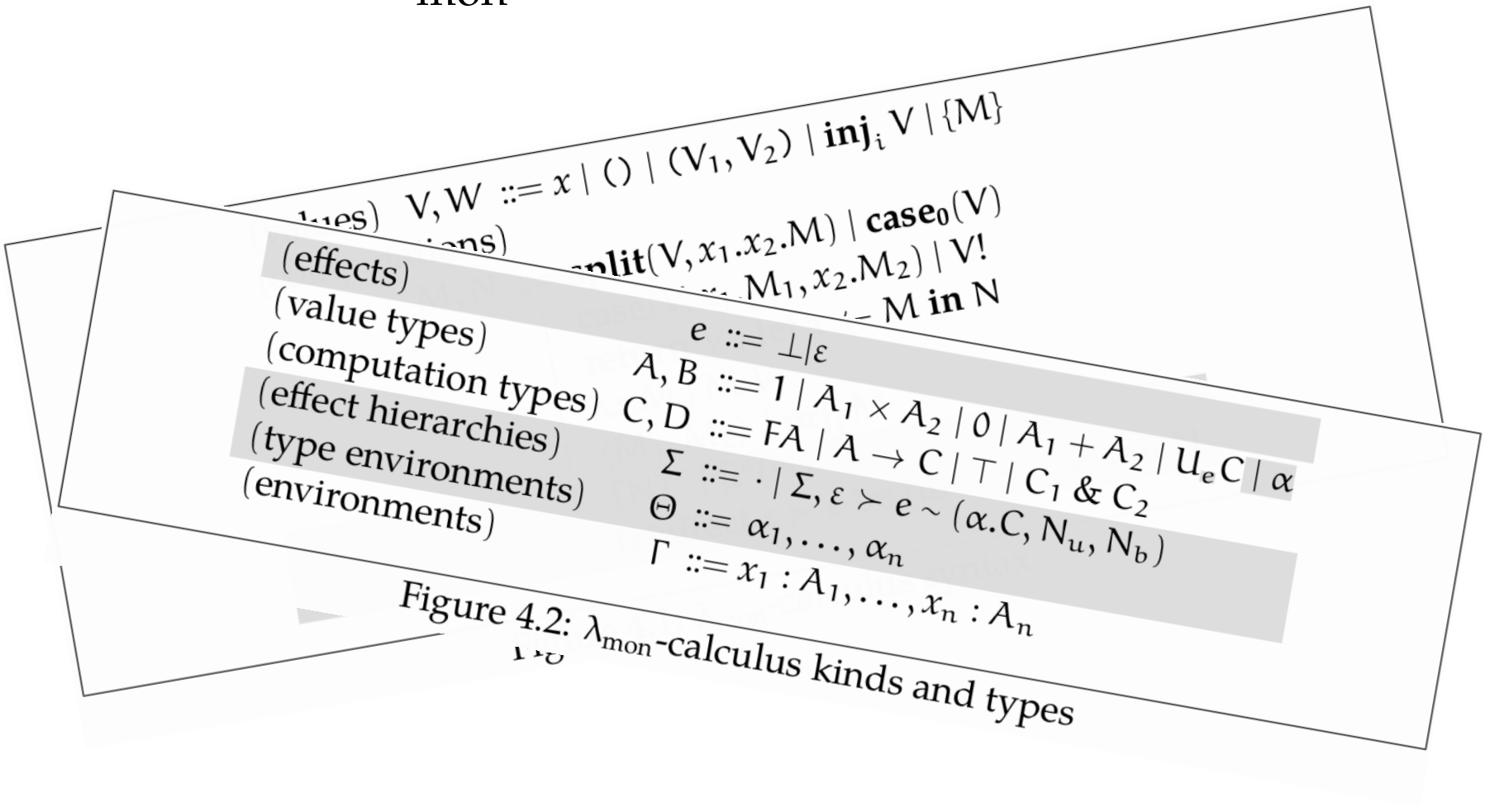


Figure 4.2: λ_{mon}-calculus kinds and types

The calculus λ_{mon}

Effect hierarchy kinding $\boxed{\Theta \vdash_k \Sigma : \mathbf{Eff}}$

$$\frac{}{\Theta \vdash_k \cdot : \mathbf{Eff}} \qquad \frac{\Theta, \alpha \vdash_k C : \mathbf{Comp}_e \quad \varepsilon \notin \Sigma \quad \Theta \vdash_k \Sigma : \mathbf{Eff}}{\Theta \vdash_k \Sigma, \varepsilon \succ e \sim (\alpha.C, N_u, N_b) : \mathbf{Eff}}$$

Value kinding $\boxed{\Theta \mid \Sigma \vdash_k A : \mathbf{Val}}$ for $\Theta \vdash_k \Sigma : \mathbf{Eff}$

$$\frac{}{\Theta \mid \Sigma \vdash_k 1 : \mathbf{Val}} \qquad \frac{\Theta \mid \Sigma \vdash_k A_1 : \mathbf{Val} \quad \Theta \mid \Sigma \vdash_k A_1 : \mathbf{Val}}{\Theta \mid \Sigma \vdash_k A_1 \times A_2 : \mathbf{Val}} \qquad \frac{}{\Theta \mid \Sigma \vdash_k 0 : \mathbf{Val}}$$

$$\frac{\Theta \mid \Sigma \vdash_k A_1 : \mathbf{Val} \quad \Theta \mid \Sigma \vdash_k A_1 : \mathbf{Val}}{\Theta \mid \Sigma \vdash_k A_1 + A_2 : \mathbf{Val}} \qquad \frac{\Theta \mid \Sigma \vdash_k e : \mathbf{Eff} \quad \Theta \mid \Sigma \vdash_k C : \mathbf{Comp}_e}{\Theta \mid \Sigma \vdash_k U_e C : \mathbf{Val}}$$

$$\frac{\alpha \in \Theta}{\Theta \mid \Sigma \vdash_k \alpha : \mathbf{Val}}$$

Effect kinding $\boxed{\Sigma \vdash_k e : \mathbf{Eff}}$ for $\Theta \vdash_k \Sigma : \mathbf{Eff}$

$$\frac{}{\Sigma \vdash_k \bot : \mathbf{Eff}} \qquad \frac{\varepsilon \in \Sigma}{\Sigma \vdash_k \varepsilon : \mathbf{Eff}}$$

Computation kinding $\boxed{\Theta \mid \Sigma \vdash_k C : \mathbf{Comp}_e}$ for $\Theta \vdash_k \Sigma : \mathbf{Eff}$ and $\Sigma \vdash_k e : \mathbf{Eff}$

$$\frac{}{\Theta \mid \Sigma \vdash_k FA : \mathbf{Comp}_e} \qquad \frac{\Theta \mid \Sigma \vdash_k A : \mathbf{Val} \quad \Theta \mid \Sigma \vdash_k C : \mathbf{Comp}_e}{\Theta \mid \Sigma \vdash_k A \to C : \mathbf{Comp}_e}$$

$$\frac{}{\Theta \mid \Sigma \vdash_k \top : \mathbf{Comp}_e} \qquad \frac{\Theta \mid \Sigma \vdash_k C_1 : \mathbf{Comp}_e \quad \Theta \mid \Sigma \vdash_k C_2 : \mathbf{Comp}_e}{\Theta \mid \Sigma \vdash_k C_1 \& C_2 : \mathbf{Comp}_e}$$

Context kinding $\boxed{\Theta \mid \Sigma \vdash_k \Gamma : \mathbf{Ctxt}}$ for $\Theta \vdash_k \Sigma : \mathbf{Eff}$

$$\frac{\text{for all } x \in \mathrm{Dom}\,(\Gamma)\text{: } \Theta \mid \Sigma \vdash_k \Gamma(x) : \mathbf{Val}}{\Theta \mid \Sigma \vdash_k \Gamma : \mathbf{Ctxt}}$$

Figure 4.3: λ_{mon}-calculus kind system

$\mathbf{inj}_i\, V \mid \{M\}$

$\mathbf{case}_0(V)$

$M_2) \mid V!$

$- M \mathbf{in} N$

$\times A_2 \mid 0 \mid A_1 + A_2 \mid U_e C \mid \alpha$

$\to C \mid \top \mid C_1 \,\&\, C_2$

$e \sim (\alpha.C, N_u, N_b)$

$_n$

$\cdot, x_n : A_n$

s and types

The calculus λ_{mon}

Effect hierarchy kinding $\boxed{\Theta \vdash_k \Sigma : \mathbf{Eff}}$

$$\frac{}{\Theta \vdash_k \cdot : \mathbf{Eff}} \qquad \frac{\Theta, \alpha \vdash_k C : \mathbf{Comp}_e}{\Theta \vdash_k \Sigma, \varepsilon \succ e}$$

Value kinding $\boxed{\Theta \mid \Sigma \vdash_k A : \mathbf{Val}}$ for $\Theta \vdash_k \Sigma : \mathbf{Eff}$

$$\frac{}{\Theta \mid \Sigma \vdash_k 1 : \mathbf{Val}} \qquad \frac{\Theta \mid \Sigma \vdash_k A_1 : \mathbf{Val} \qquad \Theta \mid \Sigma}{\Theta \mid \Sigma \vdash_k A_1 \times A_2 :}$$

$$\frac{\Theta \mid \Sigma \vdash_k A_1 : \mathbf{Val} \qquad \Theta \mid \Sigma \vdash_k A_1 : \mathbf{Val}}{\Theta \mid \Sigma \vdash_k A_1 + A_2 : \mathbf{Val}} \qquad \frac{\Theta \mid \Sigma}{}$$

$$\frac{\alpha \in \Theta}{\Theta \mid \Sigma \vdash_k \alpha :}$$

Effect kinding $\boxed{\Sigma \vdash_k e : \mathbf{Eff}}$ for $\Theta \vdash_k \Sigma : \mathbf{Eff}$

$$\frac{}{\Sigma \vdash_k \bot : \mathbf{Eff}}$$

Computation kinding $\boxed{\Theta \mid \Sigma \vdash_k C : \mathbf{Comp}_e}$

$$\frac{\Theta \mid \Sigma}{}$$

$$\frac{}{\Theta \mid \Sigma \vdash_k FA : \mathbf{Comp}_e} \qquad \frac{\Theta \mid \Sigma \vdash_k C_1 : C}{\Theta \mid}$$

$$\frac{}{\Theta \mid \Sigma \vdash_k \top : \mathbf{Comp}_e}$$

Context kinding $\boxed{\Theta \mid \Sigma \vdash_k \Gamma : \mathbf{Ctxt}}$ fc

$$\frac{\text{for all } x \in \mathrm{Dom}\,(\Gamma)\text{: } \Theta \mid \Sigma \vdash_k \ldots}{\Theta \mid \Sigma \vdash_k \Gamma : \mathbf{Ctxt}}$$

Figure 4.3: λ_{mon}-calculus kind system

Value typing $\boxed{\Theta \mid \Gamma \vdash_\Sigma V : A}$ for $\Theta \vdash_k \Sigma : \mathbf{Eff}$, $\Theta \mid \Sigma \vdash_k \Gamma : \mathbf{Ctxt}$, and $\Theta \mid \Sigma \vdash_k A : \mathbf{Val}$

$$\frac{(x : A) \in \Gamma}{\Theta \mid \Gamma \vdash_\Sigma x : A} \qquad \frac{}{\Theta \mid \Gamma \vdash_\Sigma () : 1} \qquad \frac{\Theta \mid \Gamma \vdash_\Sigma V_1 : A_1 \qquad \Theta \mid \Gamma \vdash_\Sigma V_2 : A_2}{\Theta \mid \Gamma \vdash_\Sigma (V_1, V_2) : A_1 \times A_2}$$

$$\frac{\Theta \mid \Gamma \vdash_\Sigma V : A_i}{\Theta \mid \Gamma \vdash_\Sigma \mathbf{inj}_i\, V : A_1 + A_2} \qquad \frac{\Theta \mid \Gamma \vdash_{\Sigma,e} M : C}{\Theta \mid \Gamma \vdash_\Sigma \{M\} : U_e C}$$

Computation typing $\boxed{\Theta \mid \Gamma \vdash_{\Sigma,e} M : C}$ for $\Theta \vdash_k \Sigma : \mathbf{Eff}$, $\Sigma \vdash_k e : \mathbf{Eff}$, $\Theta \mid \Sigma \vdash_k \Gamma :$ $\mathbf{Ctxt}$, and $\Theta \mid \Sigma \vdash_k C : \mathbf{Comp}_e$

$$\frac{\Theta \mid \Gamma \vdash_\Sigma V : A_1 \times A_2 \qquad \Theta \mid \Gamma, x_1 : A_1, x_2 : A_2 \vdash_{\Sigma,e} M : C}{\Theta \mid \Gamma \vdash_{\Sigma,e} \mathbf{split}(V, x_1.x_2.M) : C} \qquad \frac{\Theta \mid \Gamma \vdash_\Sigma V : 0}{\Theta \mid \Gamma \vdash_{\Sigma,e} \mathbf{case}_0(V) : C}$$

$$\frac{\Theta \mid \Gamma \vdash_\Sigma V : A_1 + A_2 \qquad \Theta \mid \Gamma, x_1 : A_1 \vdash_{\Sigma,e} M_1 : C \qquad \Theta \mid \Gamma, x_2 : A_2 \vdash_{\Sigma,e} M_2 : C}{\Theta \mid \Gamma \vdash_{\Sigma,e} \mathbf{case}(V, x_1.M_1, x_2.M_2) : C}$$

$$\frac{\Theta \mid \Gamma \vdash_\Sigma V : U_e C}{\Theta \mid \Gamma \vdash_{\Sigma,e} V! : C} \qquad \frac{\Theta \mid \Gamma \vdash_\Sigma V : A}{\Theta \mid \Gamma \vdash_{\Sigma,e} \mathbf{return}\, V : FA}$$

$$\frac{\Theta \mid \Gamma \vdash_{\Sigma,e} M : FA \qquad \Theta \mid \Gamma, x : A \vdash_{\Sigma,e} N : C}{\Theta \mid \Gamma \vdash_{\Sigma,e} \mathbf{let}\, x \leftarrow M\, \mathbf{in}\, N : C} \qquad \frac{\Theta \mid \Gamma, x : A \vdash_{\Sigma,e} M : C}{\Theta \mid \Gamma \vdash_{\Sigma,e} \lambda x.M : A \to C}$$

$$\frac{\Theta \mid \Gamma \vdash_{\Sigma,e} M : A \to C \qquad \Theta \mid \Gamma \vdash_\Sigma V : A}{\Theta \mid \Gamma \vdash_{\Sigma,e} M\, V : C}$$

$$\frac{\Theta \mid \Gamma \vdash_{\Sigma,e} M_1 : C_1 \qquad \Theta \mid \Gamma \vdash_{\Sigma,e} M_2 : C_2}{\Theta \mid \Gamma \vdash_{\Sigma,e} \langle M_1, M_2 \rangle : C_1 \,\&\, C_2} \qquad \frac{}{\Theta \mid \Gamma \vdash_{\Sigma,e} \langle\rangle : \top}$$

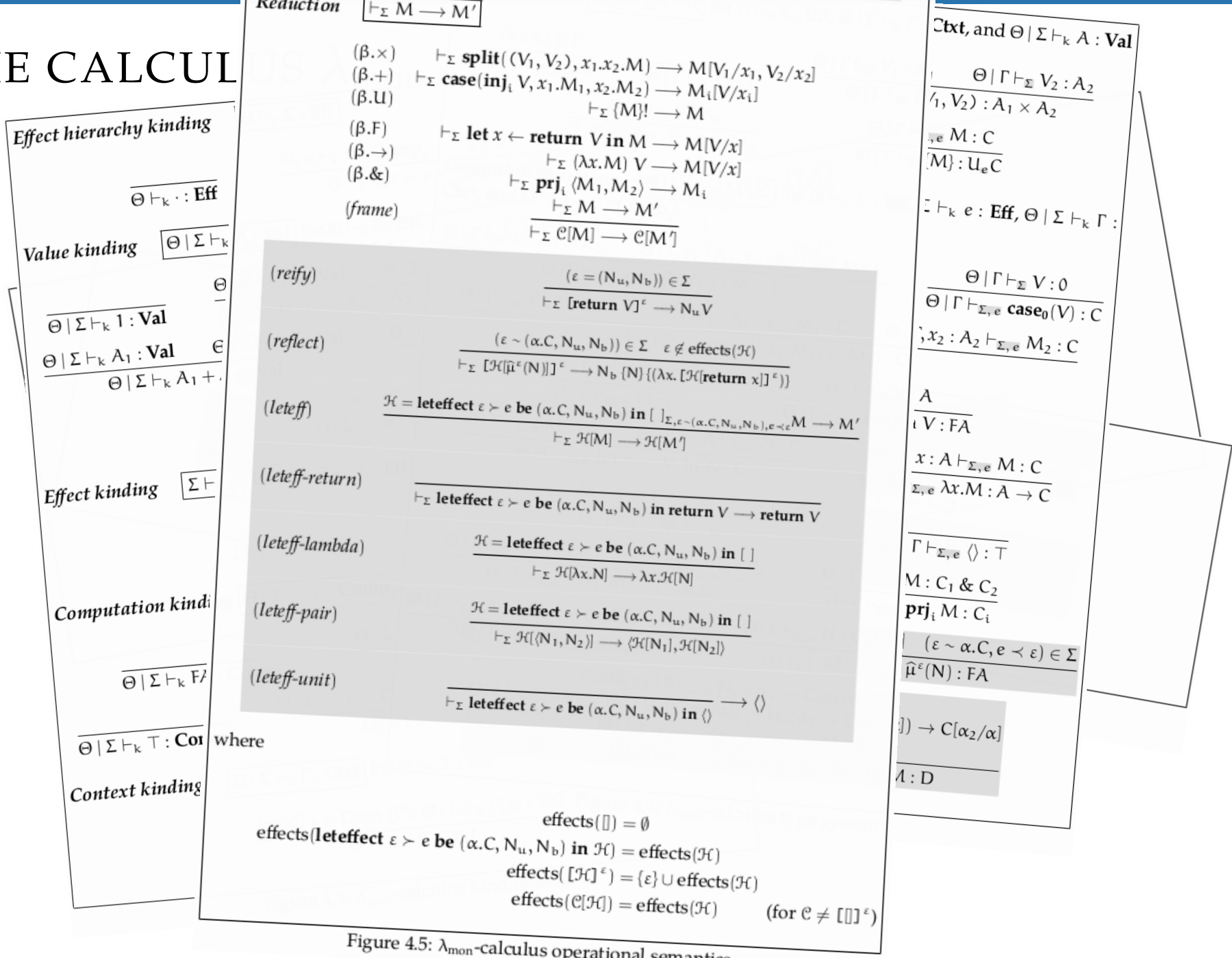
$$\frac{\Theta \mid \Gamma \vdash_{\Sigma,e} M : C_1 \,\&\, C_2}{\Theta \mid \Gamma \vdash_{\Sigma,e} \mathbf{prj}_i\, M : C_i}$$

$$\frac{\Theta \mid \Gamma \vdash_{\Sigma,\varepsilon} N : FA \qquad (\varepsilon \sim \alpha.C, e \prec \varepsilon) \in \Sigma}{\Theta \mid \Gamma \vdash_{\Sigma,e} [N]^{\varepsilon} : C[A/\alpha]} \qquad \frac{\Theta \mid \Gamma \vdash_{\Sigma,e} N : C[A/\alpha] \qquad (\varepsilon \sim \alpha.C, e \prec \varepsilon) \in \Sigma}{\Theta \mid \Gamma \vdash_{\Sigma,\varepsilon} \widehat{\mu}^{\varepsilon}(N) : FA}$$

$$\frac{\begin{array}{c}\Theta, \alpha_1 \mid \emptyset \vdash_{\Sigma,e} N_u : \alpha_1 \to C[\alpha_1/\alpha] \\ \Theta, \alpha_1, \alpha_2 \mid \emptyset \vdash_{\Sigma,e} N_b : U_e(C[\alpha_1/\alpha]) \to U_e(\alpha_1 \to C[\alpha_2/\alpha]) \to C[\alpha_2/\alpha] \\ \Theta \mid \Gamma \vdash_{(\Sigma, \varepsilon \sim (\alpha.C, N_u, N_b), e \prec \varepsilon), e} M : D\end{array}}{\Theta \mid \Gamma \vdash_{\Sigma,e} \mathbf{leteffect}\, \varepsilon \succ e\, \mathbf{be}\, (\alpha.C, N_u, N_b)\, \mathbf{in}\, M : D}$$

Figure 4.4: λ_{mon}-calculus type system

The calcul



(computation frames) $\mathcal{C} ::= \mathbf{let}\ x \leftarrow [\,]\ \mathbf{in}\ N \mid [\,]\ V \mid \mathbf{prj}_i\ [\,] \mid [\![\,]\!]^{\varepsilon}$

(hoisting contexts) $\mathcal{H} ::= [\,] \mid \mathbf{leteffect}\ \varepsilon \succ e\ \mathbf{be}\ (\alpha.C, N_u, N_b)\ \mathbf{in}\ \mathcal{H} \mid \mathcal{C}[\mathcal{H}]$

Reduction $\vdash_\Sigma M \longrightarrow M'$

$$(\beta.\times)\quad \vdash_\Sigma \mathbf{split}((V_1, V_2), x_1.x_2.M) \longrightarrow M[V_1/x_1, V_2/x_2]$$
$$(\beta.+)\quad \vdash_\Sigma \mathbf{case}(\mathbf{inj}_i\ V, x_1.M_1, x_2.M_2) \longrightarrow M_i[V/x_i]$$
$$(\beta.U)\quad \vdash_\Sigma \{M\}! \longrightarrow M$$
$$(\beta.F)\quad \vdash_\Sigma \mathbf{let}\ x \leftarrow \mathbf{return}\ V\ \mathbf{in}\ M \longrightarrow M[V/x]$$
$$(\beta.\rightarrow)\quad \vdash_\Sigma (\lambda x.M)\ V \longrightarrow M[V/x]$$
$$(\beta.\&)\quad \vdash_\Sigma \mathbf{prj}_i\ \langle M_1, M_2\rangle \longrightarrow M_i$$
$$(\mathit{frame})\quad \frac{\vdash_\Sigma M \longrightarrow M'}{\vdash_\Sigma \mathcal{C}[M] \longrightarrow \mathcal{C}[M']}$$

$$(\mathit{reify})\quad \frac{(\varepsilon = (N_u, N_b)) \in \Sigma}{\vdash_\Sigma [\mathbf{return}\ V]^{\varepsilon} \longrightarrow N_u V}$$

$$(\mathit{reflect})\quad \frac{(\varepsilon \sim (\alpha.C, N_u, N_b)) \in \Sigma \quad \varepsilon \notin \mathrm{effects}(\mathcal{H})}{\vdash_\Sigma [\mathcal{H}[\widehat{\mu}^{\varepsilon}(N)]]^{\varepsilon} \longrightarrow N_b\,\{N\}\,\{(\lambda x.\,[\mathcal{H}[\mathbf{return}\ x]]^{\varepsilon})\}}$$

$$(\mathit{leteff})\quad \frac{\mathcal{H} = \mathbf{leteffect}\ \varepsilon \succ e\ \mathbf{be}\ (\alpha.C, N_u, N_b)\ \mathbf{in}\ [\ ] \qquad [\ ]_{\Sigma, \varepsilon\sim(\alpha.C,N_u,N_b), e\prec\varepsilon} M \longrightarrow M'}{\vdash_\Sigma \mathcal{H}[M] \longrightarrow \mathcal{H}[M']}$$

$$(\mathit{leteff\text{-}return})\quad \vdash_\Sigma \mathbf{leteffect}\ \varepsilon \succ e\ \mathbf{be}\ (\alpha.C, N_u, N_b)\ \mathbf{in}\ \mathbf{return}\ V \longrightarrow \mathbf{return}\ V$$

$$(\mathit{leteff\text{-}lambda})\quad \frac{\mathcal{H} = \mathbf{leteffect}\ \varepsilon \succ e\ \mathbf{be}\ (\alpha.C, N_u, N_b)\ \mathbf{in}\ [\ ]}{\vdash_\Sigma \mathcal{H}[\lambda x.N] \longrightarrow \lambda x.\mathcal{H}[N]}$$

$$(\mathit{leteff\text{-}pair})\quad \frac{\mathcal{H} = \mathbf{leteffect}\ \varepsilon \succ e\ \mathbf{be}\ (\alpha.C, N_u, N_b)\ \mathbf{in}\ [\ ]}{\vdash_\Sigma \mathcal{H}[\langle N_1, N_2\rangle] \longrightarrow \langle \mathcal{H}[N_1], \mathcal{H}[N_2]\rangle}$$

$$(\mathit{leteff\text{-}unit})\quad \vdash_\Sigma \mathbf{leteffect}\ \varepsilon \succ e\ \mathbf{be}\ (\alpha.C, N_u, N_b)\ \mathbf{in}\ \langle\rangle \longrightarrow \langle\rangle$$

where

$$\mathrm{effects}([\,]) = \emptyset$$
$$\mathrm{effects}(\mathbf{leteffect}\ \varepsilon \succ e\ \mathbf{be}\ (\alpha.C, N_u, N_b)\ \mathbf{in}\ \mathcal{H}) = \mathrm{effects}(\mathcal{H})$$
$$\mathrm{effects}([\mathcal{H}]^{\varepsilon}) = \{\varepsilon\} \cup \mathrm{effects}(\mathcal{H})$$
$$\mathrm{effects}(\mathcal{C}[\mathcal{H}]) = \mathrm{effects}(\mathcal{H}) \qquad (\text{for } \mathcal{C} \neq [\![\,]\!]^{\varepsilon})$$

Figure 4.5: λ_{mon}-calculus operational semantics

Programming with monadic reflection

$$C^{\mathrm{ex}} := \alpha + \mathrm{string}$$
$$N_u^{\mathrm{ex}} := \lambda x.\mathbf{return}\ \mathbf{inj}_1\, x$$
$$N_b^{\mathrm{ex}} := \lambda x f.\mathbf{let}\, y \leftarrow x\ \mathbf{in}\ \mathbf{case}(y, x.fx, s.\mathbf{return}\ (\mathbf{inj}_2\, s)))$$
$$\mathrm{ex} \succ \bot \sim (\alpha.C^{\mathrm{ex}}, N_u^{\mathrm{ex}}, N_b^{\mathrm{ex}})$$

$$\mathrm{raise} := \lambda s.\hat{\mu}^{\mathrm{ex}}(\mathbf{return}\ (\mathbf{inj}_2\, s))$$
$$\mathrm{try} := \lambda c\, h.\mathbf{let}\, s \leftarrow [\,c\,]^{\mathrm{ex}}\ \mathbf{in}\ \mathbf{case}(s, a.\mathbf{return}\ a, x.hx)$$

Denotational semantics

- Denotational semantics for CBPV by Levy
- Denotational semantics for λ_{eff} by Kammar et al.
- Denotational semantics for λ_{mon} contributed

Denotational semant

The semantics assigns to each well kinded:

- value type A a set $[\![A]\!]$;
- computation type C a T-algebra $[\![C]\!]$;
- context $\vdash_k \Gamma$: **Ctxt** a set $[\![\Gamma]\!]$; and

Value types

$$[\![1]\!] := \{\star\} \qquad [\![A_1 \times A_2]\!] := [\![A_1]\!] \times [\![A_2]\!] \qquad [\![0]\!] := \emptyset$$
$$[\![A_1 + A_2]\!] := \{1\} \times [\![A_1]\!] \cup \{2\} \times [\![A_2]\!] \qquad [\![UC]\!] := |[\![C]\!]|$$

Computation types

$$[\![FA]\!] := F[\![A]\!] \qquad [\![A \to C]\!] := \langle |[\![C]\!]|^{[\![A]\!]}, \lambda f_s.\lambda x.c(\mathbf{fmap}\ (\lambda f.f(x))\ f_s)\rangle$$
$$[\![\top]\!] := \langle \{\star\}, \lambda x s.\star\rangle$$
$$[\![C_1 \,\&\, C_2]\!] := \langle |[\![C_1]\!]| \times |[\![C_2]\!]|, \lambda c_s.\langle c_1(\mathbf{fmap}\ \pi_1\ c_s), c_2(\mathbf{fmap}\ \pi_2\ c_s)\rangle\rangle$$

Contexts $[\![\Gamma]\!] := \prod_{x \in Dom(\Gamma)} [\![\Gamma(x)]\!]$

Figure 2.5: CBPV denotational semantics for types

- Denotational semantics for λ_{gg} by Levy
- Denotational semantics for λ_{gg} by Kammar et al.
- Denotational semantics for CBPV by Levy

Denot

Value terms

$$[\![x]\!](\gamma) := \pi_x(\gamma) \qquad [\![()]\!](\gamma) := \star$$

$$[\![\mathbf{inj}_i\ V]\!](\gamma) := \langle i, [\![V]\!](\gamma)\rangle \qquad [\![(V_1, V_2)]\!](\gamma) := \langle [\![V_1]\!](\gamma), [\![V_2]\!](\gamma)\rangle$$

$$[\![\{M\}]\!](\gamma) := [\![M]\!](\gamma)$$

Computation terms

$$[\![\mathbf{split}(V, x_1.x_2.M)]\!](\gamma) := [\![M]\!](\gamma[x_1 \mapsto a_1, x_2 \mapsto a_2]), \text{ where } [\![V]\!](\gamma) = \langle a_1, a_2\rangle$$

$[\![\mathbf{case}_0(V)]\!]$ is the empty map, as $[\![V]\!] : [\![\Gamma]\!] \to \emptyset$ necessitates $[\![\Gamma]\!] = \emptyset$

$$[\![\mathbf{case}(V, x_1.M_1, x_2.M_2)]\!] := \begin{cases} [\![M_1]\!](\gamma[x_1 \mapsto a_1]) & [\![V]\!](\gamma) = \langle 1, a_1\rangle \\ [\![M_2]\!](\gamma[x_2 \mapsto a_2]) & [\![V]\!](\gamma) = \langle 2, a_2\rangle \end{cases}$$

$$[\![V!]\!](\gamma) := [\![V]\!](\gamma) \qquad [\![\mathbf{return}\ V]\!](\gamma) := \mathbf{return}\ ([\![V]\!](\gamma))$$

$$[\![\mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N]\!](\gamma) := [\![M]\!](\gamma) \gg\!= \lambda a.\, [\![N]\!](\gamma[x \mapsto a])$$

$$[\![\lambda x.M]\!](\gamma) := \lambda a.\, [\![M]\!](\gamma[x \mapsto a]) \qquad [\![M\ V]\!](\gamma) := ([\![M]\!](\gamma))([\![V]\!](\gamma))$$

$$[\![\langle M_1, M_2\rangle]\!](\gamma) := \langle [\![M_1]\!](\gamma), [\![M_2]\!](\gamma)\rangle \qquad [\![\mathbf{prj}_i\ M]\!](\gamma) := \pi_i([\![M]\!](\gamma)) \qquad [\![\langle\rangle]\!](\gamma) := \star$$

Contexts $[\![\Gamma]\!] := \prod_{x \in \mathrm{Dom}(\Gamma)} [\![$

Figure 2.6: CBPV denotational semantics for terms

Figure 2.5: CBPV den

Denot

Value terms

$$[\![x]\!](\gamma) := \pi_x(\gamma) \qquad [\![()]\!](\gamma) := \star$$
$$[\![\mathbf{inj}_i\ V]\!](\gamma) := \langle i, [\![V]\!](\gamma)\rangle \qquad [\![(V_1, V_2)]\!](\gamma) := \langle [\![V_1]\!](\gamma), [\![V_2]\!](\gamma)\rangle$$
$$[\![\{M\}]\!](\gamma) := [\![M]\!](\gamma)$$

Computation terms

$$[\![\mathbf{split}(V, x_1.x_2.M)]\!](\gamma) := [\![M]\!](\gamma[x_1 \mapsto a_1, x_2 \mapsto a_2]), \text{ where } [\![V]\!](\gamma) = \langle a_1, a_2\rangle$$
$[\![\mathbf{case}_0(V)]\!]$ is the empty map, as $[\![V]\!] : [\![\Gamma]\!] \to \emptyset$ necessitates $[\![\Gamma]\!] = \emptyset$

- value type $\vdash_k A$: **Val** a set $[\![A]\!]$;
- effect $\vdash_k E$: **Eff** a (parameterised) signature $[\![E]\!]$;
- computation type $\vdash_k C$: $\mathbf{Comp}_E$ a $[\![E]\!]$-algebra $[\![C]\!]$;
- context $\vdash_k \Gamma$: **Ctxt** a set $[\![\Gamma]\!]$; and
- handler $\vdash_k X$: **Hndlr** an algebra and a function into the carrier of that algebra.

$$[\![\lambda x.M]\!](\gamma) := \lambda a.\, [\![M]\!](\gamma[x \mapsto a]) \qquad [\![\mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N]\!](\gamma) := [\![M]\!](\gamma) \gg\!= \lambda a.\, [\![N]\!](\gamma[x \mapsto a])$$
$$[\![\langle M_1, M_2\rangle]\!](\gamma) := \langle [\![M_1]\!](\gamma), [\![M_2]\!](\gamma)\rangle \qquad [\![M\ V]\!](\gamma) := ([\![M]\!](\gamma))([\![V]\!](\gamma)) \qquad [\![\mathbf{prj}_i\ M]\!](\gamma) := \pi_i([\![M]\!](\gamma))$$

Figure 2.6: CBPV denotational semantics for terms

Contexts $[\![\Gamma]\!] := \prod_{x \in \mathrm{Dom}(\Gamma)} [\![$

Figure 2.5: CBPV den

Denot

Value terms

$$[\![x]\!](\gamma) := \pi_x(\gamma) \qquad [\![()]\!](\gamma) := \star$$
$$[\![\mathbf{inj}_i\ V]\!](\gamma) := \langle i, [\![V]\!](\gamma)\rangle \qquad [\![(V_1, V_2)]\!](\gamma) := \langle [\![V_1]\!](\gamma), [\![V_2]\!](\gamma)\rangle$$
$$[\![\{M\}]\!](\gamma) := [\![M]\!](\gamma)$$

Computation terms

$$[\![\mathbf{split}(V, x_1.x_2.M)]\!](\gamma) := [\![M]\!](\gamma[x_1 \mapsto a_1, x_2 \mapsto a_2]), \text{ where } [\![V]\!](\gamma) = \langle a_1, a_2\rangle$$
$[\![\mathbf{case}_0(V)]\!]$ is the empty map, as $[\![V]\!] : [\![\Gamma]\!] \to \emptyset$ necessitates $[\![\Gamma]\!] = \emptyset$

- value type $\vdash_k A$: **Val** a set $[\![A]\!]$;
- effect $\vdash_k E$: **Eff** a (parameterised) signature $[\![E]\!]$;
- computation type $\vdash_k C$: $\mathbf{Comp}_E$ a $[\![E]\!]$-algebra $[\![C]\!]$;
- context $\vdash_k \Gamma$: **Ctxt** a set $[\![\Gamma]\!]$; and
- handler $\vdash_k X$: **Hndlr** an algebra and a function into the carrier of that algebra.

$$[\![A \xrightarrow{E} C]\!] := \sum_{[\![E]\!]\text{-algebras with carrier } |[\![C]\!]|} |[\![C]\!]|^{|[\![A]\!]|}$$

$$[\![\lambda x.M]\!](\gamma) := \lambda a.\, [\![M]\!](\gamma[x \mapsto a]) \qquad [\![\mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N]\!](\gamma) := [\![M]\!](\gamma) \gg\!= \lambda a.\, [\![N]\!](\gamma[x \mapsto a])$$
$$[\![\langle M_1, M_2\rangle]\!](\gamma) := \langle [\![M_1]\!](\gamma), [\![M_2]\!](\gamma)\rangle \qquad [\![M\ V]\!](\gamma) := ([\![M]\!](\gamma))([\![V]\!](\gamma)) \qquad [\![\mathbf{prj}_i\ M]\!](\gamma) := \pi_i([\![M]\!](\gamma))$$
$$[\![()]\!] \qquad [\![\mathbf{return}\ (V)]\!](\gamma) := \mathbf{return}\ ([\![V]\!](\gamma)) \qquad [\![\langle\rangle]\!](\gamma) := \star$$

Figure 2.6: CBPV denotational semantics for terms

Contexts $[\![\Gamma]\!] := \prod_{x \in \mathrm{Dom}(\Gamma)} [\![$

Figure 2.5: CBPV den

Denot

Value terms

$$[\![x]\!](\gamma) := \pi_x(\gamma) \qquad [\![()]\!](\gamma) := \star$$
$$[\![\mathbf{inj}_i\ V]\!](\gamma) := \langle i, [\![V]\!](\gamma)\rangle \qquad [\![(V_1, V_2)]\!](\gamma) := \langle [\![V_1]\!](\gamma), [\![V_2]\!](\gamma)\rangle$$
$$[\![\{M\}]\!](\gamma) := [\![M]\!](\gamma)$$

Computation terms

$$[\![\mathbf{split}(V, x_1.x_2.M)]\!](\gamma) := [\![M]\!](\gamma[x_1 \mapsto a_1, x_2 \mapsto a_2]), \text{ where } [\![V]\!](\gamma) = \langle a_1, a_2\rangle$$
$[\![\mathbf{case}_0(V)]\!]$ is the empty map, as $[\![V]\!] : [\![\Gamma]\!] \to \emptyset$ necessi
$[\![\emptyset]\!] := \langle \emptyset, i\rangle$
$[\![A \ ^{E}\!\Rightarrow^{E'} C]\!] :=$
$[\![\mathbf{let}\ x$
$[\![\lambda x.M]\!](\gamma) :=$
$[\![\{\mathrm{op} : A \to B\} \uplus E]\!] := \langle \{\mathrm{op}\} \cup [\![E]\!], \mathrm{arity}_{[\![E]\!]}[\mathrm{op} \mapsto \langle [\![A]\!], [\![B]\!]\rangle]\rangle$
$[\![V]\!](\gamma) = \langle 1, a_1\rangle$
$[\![V]\!](\gamma) = \langle 2, a_2\rangle$
of that algebra. $(\gamma) := \mathbf{return}\ ([\![V]\!](\gamma))$
$[\![M\ V]\!](\gamma) := ([\![M]\!](\gamma))([\![V]\!](\gamma))$
$[\![\mathbf{prj}_i\ M]\!](\gamma) := \pi_i([\![M]\!](\gamma))$
$[\![\langle\rangle]\!](\gamma) := \star$

Contexts $[\![\Gamma]\!] := \prod_{x \in \mathrm{Dom}(\Gamma)}$

- value type $\vdash_k A$: **Val** a set $[\![A]\!]$;
- effect $\vdash_k E$: **Eff** a (parameterised) signature $[\![E]\!]$;
- computation type $\vdash_k C$: **Comp**$_E$ a $[\![E]\!]$-algebra $[\![C]\!]$;
- context $\vdash_k \Gamma$: **Ctxt** a set $[\![\Gamma]\!]$; and
- handler $\vdash_k X$: **Hndlr** an algebra and a fu

Figure 2.6: CBPV denotational semantics for terms

Figure 2.5: CBPV den

Denot

Value terms

$$[\![x]\!](\gamma) := \pi_x(\gamma) \qquad [\![()]\!](\gamma) := \star$$
$$[\![\mathbf{inj}_i\ V]\!](\gamma) := \langle i, [\![V]\!](\gamma)\rangle \qquad [\![(V_1, V_2)]\!](\gamma) := \langle [\![V_1]\!](\gamma), [\![V_2]\!](\gamma)\rangle$$
$$[\![\{M\}]\!](\gamma) := [\![M]\!](\gamma)$$

Computation terms

$$[\![\mathbf{split}(V, x_1.x_2.M)]\!](\gamma) := [\![M]\!](\gamma[x_1 \mapsto a_1, x_2 \mapsto a_2]), \text{ where } [\![V]\!](\gamma) = \langle a_1, a_2\rangle$$

$[\![\mathbf{case}_0(V)]\!]$ is the empty map, as $[\![V]\!] : [\![\Gamma]\!] \to \emptyset$ necessi

$$[\![\emptyset]\!] := \langle \emptyset, i\rangle$$

$$[\![\mathbf{case}(V, M_1, M_2)]\!](\gamma) := \begin{cases} [\![M_1]\!](\gamma[x_1 \mapsto a]) & \text{if } [\![V]\!](\gamma) = \langle 1, a\rangle \\ [\![M_2]\!](\gamma[x_2 \mapsto a]) & \text{if } [\![V]\!](\gamma) = \langle 2, a\rangle \end{cases}$$

$$[\![\mathbf{return}\ V]\!](\gamma) := \mathbf{return}\ ([\![V]\!](\gamma))$$

$$[\![\mathbf{let}\ x \Leftarrow M\ \mathbf{in}\ N]\!](\gamma) := [\![M]\!](\gamma) \gg\!= \lambda a.\, [\![N]\!](\gamma[x \mapsto a])$$

$$[\![M\ V]\!](\gamma) := ([\![M]\!](\gamma))([\![V]\!](\gamma)) \qquad [\![\langle\rangle]\!](\gamma) := \star$$

$$[\![\mathbf{prj}_i\ M]\!](\gamma) := \pi_i([\![M]\!](\gamma))$$

Figure 2.6: CBPV denotational semantics for terms

- value type $\vdash_k A$: **Val** a set $[\![A]\!]$;
- effect $\vdash_k E$: **Eff** a (parameterised) signature $[\![E]\!]$;
- computation type $\vdash_k C$: **Comp**$_E$ a $[\![E]\!]$-algebra $[\![C]\!]$;

$$[\![A \to^E C]\!] := \quad [\![\mathbf{op}\ V(\lambda x.M)]\!](\gamma) := \mathbf{op}_{[\![V]\!]\gamma}(\lambda b : [\![B]\!].\, [\![M]\!](\gamma[x \mapsto b]))$$

$$[\![\lambda x.M]\!](\gamma) :=$$

$$[\![\{\mathbf{op} : A \to B\} \uplus E]\!] := \langle \{\mathbf{op}\} \cup [\![E]\!], \quad \to a])$$

$$\quad := \langle [\![M_1]\!](\gamma), [\![M_2]\!](\gamma)\rangle$$

$$[\![\mathbf{let}\ x \quad := [\![M]\!](\gamma) \gg\!= \lambda a.\, [\![N]\!](\gamma[x \mapsto a])$$

$$[\![\mathbf{return}\ V]\!](\gamma) := \mathbf{return}\ ([\![V]\!](\gamma))$$

$$[\![M\ V]\!](\gamma) := ([\![M]\!](\gamma))([\![V]\!](\gamma))$$

$$[\![\mathbf{prj}_i\ M]\!](\gamma) := \pi_i([\![M]\!](\gamma))$$

Figure 2.5: CBPV den

Contexts $[\![\Gamma]\!] := \prod_{x \in \mathrm{Dom}(\Gamma)}$

DENOT

Value terms

$$[\![x]\!](\gamma) := \pi_x(\gamma) \qquad [\![()]\!](\gamma) := \star$$

$$[\![\mathbf{inj}_i\ V]\!](\gamma) := \langle i, [\![V]\!](\gamma)\rangle \qquad [\![(V_1, V_2)]\!](\gamma) := \langle [\![V_1]\!](\gamma), [\![V_2]\!](\gamma)\rangle$$

$$[\![\{M\}]\!](\gamma) := [\![M]\!](\gamma)$$

Computation terms

$$[\![\mathbf{split}(V, x_1.x_2.M)]\!](\gamma) := [\![M]\!](\gamma[x_1 \mapsto a_1, x_2 \mapsto a_2]), \text{ where } [\![V]\!](\gamma) = \langle a_1, a_2\rangle$$

$[\![\mathbf{case}_0(V)]\!]$ is the empty map, as $[\![V]\!] : [\![\Gamma]\!] \to \emptyset$ necessi

- value type $\vdash_k A$: **Val** a set $[\![A]\!]$;
- effect $\vdash_k E$: **Eff** a (parameter
- computation typ

$[\![\mathbf{handle}\ M\ \mathbf{with}\ H]\!](\gamma) := [\![M]\!](\gamma) \gg\!= f$

algebra $[\![C]\!]$; $[\![A \xRightarrow{E} C]\!] :=$

$[\![\emptyset]\!] := \langle \emptyset, i\rangle$, $a_2\rangle$

$[\![\ ]\!](\gamma) := \mathrm{op}_{[\![V]\!]\gamma}(\lambda b : [\![B]\!].[\![M]\!](\gamma[x \mapsto b]))$ $\mapsto \langle [\![A]\!], [\![B]\!]\rangle]\rangle$ $(\gamma) = \langle 1, a_1\rangle$

$[\![\lambda x.M]\!](\gamma) :=$ $[\![\mathbf{let}\ x$ $[\![V]\!](\gamma) = \langle 2, a_2\rangle$

$[\![\{\mathrm{op} : A \to B\} \uplus E]\!] := \langle \{\mathrm{op}\} \cup [\![E$ $(\gamma) := [\![M]\!](\gamma) \gg\!= \lambda a.[\![N]\!](\gamma[x \mapsto a])$ $(\gamma) := \mathbf{return}\ ([\![V]\!](\gamma))$

$\to a])$ $[\![M\ V]\!](\gamma) := ([\![M]\!](\gamma))([\![V]\!](\gamma))$

$:= \langle [\![M_1]\!](\gamma), [\![M_2]\!](\gamma)\rangle$ $[\![\mathbf{prj}_i\ M]\!](\gamma) := \pi_i([\![M]\!](\gamma))$ $[\![\langle\rangle]\!](\gamma) := \star$

$[\![C$

Contexts $[\![\Gamma]\!] := \prod_{x \in \mathrm{Dom}(\Gamma)}$

Figure 2.6: CBPV denotational semantics for terms

Figure 2.5: CBPV den

Denot

Value terms

$$[\![x]\!](\gamma) := \pi_x(\gamma) \qquad [\![()]\!](\gamma) := \star$$
$$[\![\mathbf{inj}_i\ V]\!](\gamma) := \langle i, [\![V]\!](\gamma)\rangle \qquad [\![(V_1, V_2)]\!](\gamma) := \langle [\![V_1]\!](\gamma), [\![V_2]\!](\gamma)\rangle$$
$$[\![\{M\}]\!](\gamma) := [\![M]\!](\gamma)$$

Computation terms

$$[\![\mathbf{split}(V, x_1.x_2.M)]\!](\gamma) := [\![M]\!](\gamma[x_1 \mapsto a_1, x_2 \mapsto a_2]), \text{ where } [\![V]\!](\gamma) = \langle a_1, a_2\rangle$$
$$[\![\mathbf{case}_0(V)]\!] \text{ is the empty map, as } [\![V]\!] : [\![\Gamma]\!] \to \emptyset \text{ necessi}$$
$$[\![\emptyset]\!] := \langle \emptyset, i\rangle$$
$$[\![\mathbf{case}_{1+n}(V)]\!](\gamma) := [\![M]\!](\gamma) \gg\!= f$$
$$[\![A \to^{E} C]\!] := [\![A]\!], [\![B]\!]\rangle]\rangle$$
$$[\![V]\!](\gamma) = \langle 1, a_1\rangle$$
$$[\![V]\!](\gamma) = \langle 2, a_2\rangle$$
$$[\![\mathbf{return}\ V]\!](\gamma) := \mathbf{return}\ ([\![V]\!](\gamma))$$
$$[\![\mathbf{let}\ x \leftarrow M\ \mathbf{in}\ N]\!](\gamma) := [\![M]\!](\gamma) \gg\!= (a.[\![N]\!](\gamma[x \mapsto a]))$$
$$[\![\mathbf{force}\ V]\!](\gamma) := [\![V]\!](\gamma)$$
$$[\![\lambda x.M]\!](\gamma) := a \mapsto [\![M]\!](\gamma[x \mapsto a])$$
$$[\![M\ V]\!](\gamma) := ([\![M]\!](\gamma))([\![V]\!](\gamma))$$
$$[\![\langle\rangle]\!](\gamma) := \star$$
$$[\![\mathbf{prj}_i\ M]\!](\gamma) := \pi_i([\![M]\!](\gamma))$$

Contexts $[\![\Gamma]\!] := \prod_{x \in \mathrm{Dom}(\Gamma)} [\![$

Figure 2.6: CBPV denotational semantics for terms

- value type $\vdash_k A$: **Val** a set $[\![A]\!]$;
- effect $\vdash_k E$: **Eff** a (parameter
- computation typ

The semantics assigns to each $\theta \in [\![\Theta]\!]$ and well kinded

- value type $\Theta \mid \Sigma \vdash_k A$: **Val** a set $[\![A]\!](\theta)$;
- effect $\Sigma \vdash_k e$: **Eff** a monad $[\![e]\!]$;
- e-based computation type $\Theta \mid \Sigma \vdash_k C$: $\mathbf{Comp}_e$ a $[\![e]\!]$-algebra $[\![C]\!](\theta)$;
- context $\Theta \mid \Sigma \vdash_k \Gamma$: **Ctxt** a set of tuples $[\![\Gamma]\!](\theta)$ induced by Γ

Figure 2.5: CBPV den

Denot

Value terms

$$[\![x]\!](\gamma) := \pi_x(\gamma) \qquad [\![()]\!](\gamma) := \star$$
$$[\![\mathbf{inj}_i\ V]\!](\gamma) := \langle i, [\![V]\!](\gamma)\rangle \qquad [\![(V_1, V_2)]\!](\gamma) := \langle [\![V_1]\!](\gamma), [\![V_2]\!](\gamma)\rangle$$
$$[\![\{M\}]\!](\gamma) := [\![M]\!](\gamma)$$

Computation terms

$$[\![\mathbf{split}(V, x_1.x_2.M)]\!](\gamma) := [\![M]\!](\gamma[x_1 \mapsto a_1, x_2 \mapsto a_2]), \text{ where } [\![V]\!](\gamma)$$

$[\![\mathbf{case}_0(V)]\!]$ is the empty map, as $[\![V]\!] : [\![\Gamma]\!] \to \emptyset$ necess

$$[\![\emptyset]\!] := \langle \emptyset, i\rangle \qquad , a_2\rangle$$

$$[\![\varepsilon]\!] := \langle T_e, \mathbf{return}_e, \gg=_e\rangle$$

where

$$T_\varepsilon X := |[\![C]\!](\theta[\alpha \mapsto X])|$$
$$\mathbf{return}_\varepsilon{}^X := [\![N_u]\!](\theta[\alpha \mapsto X]) : X \to T_\varepsilon X$$
$$\gg=_\varepsilon{}^{X,Y} := [\![N_b]\!](\theta[\alpha_1 \mapsto X, \alpha_2 \mapsto Y]) : T_\varepsilon X \to (X \to T_\varepsilon Y) \to T_\varepsilon Y.$$

and define $[\![\bot]\!]$ to be the identity monad.

$$[\![\langle\rangle]\!](\gamma) := \star$$
$$[\![\quad]\!](\gamma) := \pi_i([\![M]\!](\gamma))$$

- value type $\vdash_k A$: **Val** a set $[\![A]\!]$;
- effect $\vdash_k E$: **Eff** a (parameter
- computation typ

$$(\gamma) := [\![M]\!](\gamma) \gg= f$$

$$[\![A \ ^{E}\!\Rightarrow^{E'} C]\!] := \quad [\![A]\!], [\![B]\!]\rangle]\rangle$$

The semantics assigns to each $\theta \in [\![\Theta]\!]$ and well kinded

- value type $\Theta \mid \Sigma \vdash_k A$: **Val** a set $[\![A]\!](\theta)$;
- effect $\Sigma \vdash_k e$: **Eff** a monad $[\![e]\!]$;
- e-based computation type $\Theta \mid \Sigma \vdash_k C$: **Comp**$_e$ a $[\![e]\!]$-algebra
- context $\Theta \mid \Sigma \vdash_k \Gamma$: **Ctxt** a set of tuples $[\![\Gamma]\!](\theta)$ induced by Γ

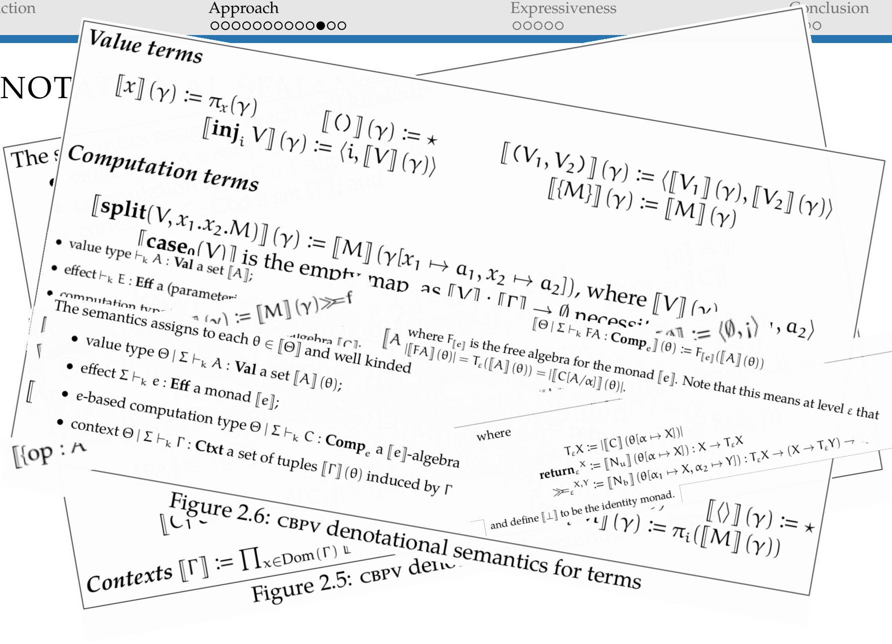
$[\![\{op : A$

Contexts $[\![\Gamma]\!] := \prod_{x \in \mathrm{Dom}(\Gamma)}$

Figure 2.6: CBPV denotational semantics for terms

Figure 2.5: CBPV den

DENOT


Value terms
$[\![x]\!](\gamma) := \pi_x(\gamma)$
$[\![()]\!](\gamma) := \star$
$[\![\mathbf{inj}_i\, V]\!](\gamma) := \langle i, [\![V]\!](\gamma)\rangle$
$[\![(V_1, V_2)]\!](\gamma) := \langle [\![V_1]\!](\gamma), [\![V_2]\!](\gamma)\rangle$
$[\![\{M\}]\!](\gamma) := [\![M]\!](\gamma)$
Computation terms
$[\![\mathbf{split}(V, x_1.x_2.M)]\!](\gamma) := [\![M]\!](\gamma[x_1 \mapsto a_1, x_2 \mapsto a_2])$, where $[\![V]\!](\gamma)$
The semantics assigns to each $\theta \in [\![\Theta]\!]$ and well kinded
value type $\Theta \mid \Sigma \vdash_\kappa A : \mathbf{Val}$ a set $[\![A]\!](\theta)$;
effect $\Sigma \vdash_\kappa e : \mathbf{Eff}$ a monad $[\![e]\!]$;
e-based computation type $\Theta \mid \Sigma \vdash_\kappa C : \mathbf{Comp}_e$ a $[\![e]\!]$-algebra
context $\Theta \mid \Sigma \vdash_\kappa \Gamma : \mathbf{Ctxt}$ a set of tuples $[\![\Gamma]\!](\theta)$ induced by Γ
where $F_{[\![e]\!]}$ is the free algebra for the monad $[\![e]\!]$. Note that this means at level ε that
Figure 2.6: CBPV denotational semantics for terms
Contexts $[\![\Gamma]\!] := \prod_{x \in \mathrm{Dom}(\Gamma)}$
Figure 2.5: CBPV deno

Denot

Value terms

$$[\![x]\!](\gamma) := \pi_x(\gamma) \qquad [\![()]\!](\gamma) := \star$$
$$[\![\mathbf{inj}_i\ V]\!](\gamma) := \langle i, [\![V]\!](\gamma)\rangle \qquad [\![(V_1, V_2)]\!](\gamma) := \langle [\![V_1]\!](\gamma), [\![V_2]\!](\gamma)\rangle$$
$$[\![\{M\}]\!](\gamma) := [\![M]\!](\gamma)$$

Computation terms

$$[\![\mathbf{split}(V, x_1.x_2.M)]\!](\gamma) := [\![M]\!](\gamma[x_1 \mapsto a_1, x_2 \mapsto a_2]), \text{ where } [\![V]\!](\gamma)$$
$[\![\mathbf{case}_0(V)]\!]$ is the empty map as $[\![V]\!] \cdot [\![\Gamma]\!] \to \emptyset$ neces
$[\![\Theta \mid \Sigma \vdash_k FA : \mathbf{Comp}_e]\!](\theta) := F_{[e]}([\![A]\!](\theta))$

- value type $\Theta \mid \Sigma \vdash_k A : \mathbf{Val}$ a set $[\![A]\!](\theta)$;
- effect $\Sigma \vdash_k e : \mathbf{Eff}$ a monad $[\![e]\!]$;
- e-based computation type $\Theta \mid \Sigma \vdash_k C : \mathbf{Comp}_e$ $[\![A$ where $F_{[e]}$ is the free algebra for the monad $[\![e]\!]$. Note that this means at level ε that $|[\![FA]\!](\theta)| = T_\varepsilon([\![A]\!](\theta)) = |[\![C[A/\alpha]]\!](\theta)|$.
- context $\Theta \mid \Sigma \vdash_k \Gamma : \mathbf{Ctxt}$ a set of tuples $[\![\Gamma]\!](\theta)$ induced by Γ

The semantics assigns to each $\theta \in [\![\Theta]\!]$ and well kinded

$$[\![\widehat{\mu}^{\varepsilon}(N)]\!](\theta)(\gamma) := [\![N]\!](\theta)(\gamma).$$

Figure 2.6: CBPV denotational semantics for terms

Contexts $[\![\Gamma]\!] := \prod_{x \in \mathrm{Dom}(\Gamma)}$

Figure 2.5: CBPV den

Denot

Value terms

$$[\![x]\!](\gamma) := \pi_x(\gamma) \qquad [\![()]\!](\gamma) := \star$$
$$[\![\mathbf{inj}_i\ V]\!](\gamma) := \langle i, [\![V]\!](\gamma)\rangle \qquad [\![(V_1, V_2)]\!](\gamma) := \langle [\![V_1]\!](\gamma), [\![V_2]\!](\gamma)\rangle$$
$$[\![\{M\}]\!](\gamma) := [\![M]\!](\gamma)$$

Computation terms

$$[\![\mathbf{split}(V, x_1.x_2.M)]\!](\gamma) := [\![M]\!](\gamma[x_1 \mapsto a_1, x_2 \mapsto a_2]), \text{ where } [\![V]\!](\gamma)$$
$[\![\mathbf{case}_0(V)]\!]$ is the empty map as $[\![V]\!] \cdot [\![\Gamma]\!] \to \emptyset$ necessi… $\langle \emptyset, i\rangle$ …, $a_2\rangle$

- value type $\vdash_k A$: **Val** a set $[\![A]\!]$;
- effect $\vdash_k E$: **Eff** a (parame…
- computation ty… $(\gamma) := [\![M]\!](\gamma) \gg\!= f$

The semantics assigns to each $\theta \in [\![\Theta]\!]$ and well kinded

- value type $\Theta \mid \Sigma \vdash_k A$: **Val** a set $[\![A]\!](\theta)$;
- effect $\Sigma \vdash_k e$: **Eff** a monad $[\![e]\!]$;
- e-based computation type $\Theta \mid \Sigma \vdash_k C$
- context $\Theta \mid \Sigma \vdash_k \Gamma$: **Ctxt** …

$[\![A$ where $F_{[\![e]\!]}$ is the free algebra for the monad $[\![$ …
$|[\![FA]\!](\theta)| = T_\varepsilon([\![A]\!](\theta)) = |[\![C[A/\alpha]]\!](\theta)|$ …t this means at level ε that

$[\![\mathbf{Comp}_e]\!](\theta) := F_{[\![e]\!]}([\![A]\!](\theta))$

$[\![\{\mathrm{op} : A$ …

$[\![\Theta \mid \Gamma \vdash_{\Sigma,e} [N]^\varepsilon : C[A/\alpha]]\!](\theta)(\gamma) : [\![\Gamma]\!](\theta) \to |[\![FA : \mathbf{Comp}_e]\!](\theta)| = |[\![C[A/\alpha]]\!](\theta)|$

$[\![[N]^\varepsilon]\!](\theta)(\gamma) := [\![N]\!](\theta)(\gamma)$ …$:= [\![N]\!](\theta)(\gamma).$

…(α ↦ A)) · A → T_ε X
…X, α₂ ↦ Y]) : T_ε X → (X → T_ε Y) →

and define $[\![\bot]\!]$ to be the id… …$(\gamma) := \star$

$[\![\pi_i]\!](\gamma) := \pi_i([\![M]\!](\gamma))$

Contexts $[\![\Gamma]\!] := \prod_{x \in \mathrm{Dom}(\Gamma)}$ …

CBPV denotational semantics for terms

Figure 2.5: CBPV den…

Denot

Value terms

$$[\![x]\!](\gamma) := \pi_x(\gamma) \qquad [\![()]\!](\gamma) := \star$$
$$[\![\mathbf{inj}_i\ V]\!](\gamma) := \langle i, [\![V]\!](\gamma)\rangle \qquad [\![(V_1, V_2)]\!](\gamma) := \langle [\![V_1]\!](\gamma), [\![V_2]\!](\gamma)\rangle$$
$$[\![\{M\}]\!](\gamma) := [\![M]\!](\gamma)$$

Computation terms

$$[\![\mathbf{split}(V, x_1.x_2.M)]\!](\gamma) := [\![M]\!](\gamma[x_1 \mapsto a_1, x_2 \mapsto a_2]), \text{ where } [\![V]\!](\gamma)$$
$[\![\mathbf{case}_0(V)]\!]$ is the empty map as $[\![V]\!] \cdot [\![\Gamma]\!] \to \emptyset$ necessi… $[\![\Theta \mid \Sigma \vdash_k FA : \mathbf{Comp}_e]\!](\theta) := \langle \emptyset, i\rangle$ …, $a_2\rangle$

- value type $\vdash_k A$: **Val** a set $[\![A]\!]$;
- effect $\vdash_k E$: **Eff** a (parameter…
- computation typ… $(\gamma) := [\![M]\!](\gamma) \gg\!= f$

The semantics assigns to each $\theta \in [\![\Theta]\!]$ and well kinded

- value type $\Theta \mid \Sigma \vdash_k A$: **Val** a set $[\![A]\!](\theta)$;

$[\![A$ where $F_{[e]}$ is the free algebra for the monad …
$|[\![FA]\!](\theta)| = T_\varepsilon([\![A]\!](\theta)) = |[\![C[A/\alpha]]\!](\theta)|$

$[\![\ [\![\Theta \mid \Gamma \vdash_{\Sigma,e} \mathbf{leteffect}\ \varepsilon \succ e\ \mathbf{be}\ (\alpha.C, N_u, N_b)\ \mathbf{in}\ N : D]\!](\theta)(\gamma) := \begin{cases} [\![N]\!](\theta)(\gamma) & \text{if } N_u \text{ and } N_b \text{ form a monad} \\ \text{undefined} & \text{otherwise} \end{cases}$

$\mathbf{Comp}_e]\!](\theta)| = |[\![C[A/\alpha]]\!](\theta)|$ …t this means at level ε that

$[\![\{\mathrm{op} : A$ … $\cdot \to T_\varepsilon X$ … $X, \alpha_2 \mapsto Y]) : T_\varepsilon X \to (X \to T_\varepsilon Y) \to$

$[\![\Theta \mid \Gamma \vdash_{\Sigma,e} [N]^\varepsilon : C[A/\alpha]]\!]$ …

$[\![[N]^\varepsilon]\!](\theta)(\gamma) := [\![N]\!](\theta)(\gamma)$

and den…

Contexts $[\![\Gamma]\!] := \prod_{x \in \mathrm{Dom}(\Gamma)}$ …

CBPV denotational semantics for terms

Figure 2.5: CBPV den…

Adequacy and Soundness

Theorem 2.18 (adequacy)**.** *Denotational equivalence implies contextural equivalence. Explicitly: Given a monad satisfying the mono requirement, then for all* $\Gamma \vdash P, Q : X$*, if* $[\![P]\!] = [\![Q]\!]$ *then* $P \simeq Q$*.*

Adequacy and Soundness

Theorem 2.18 (adequacy)**.** *Denotational equivalence implies contextural equivalence. Explicitly: Given a monad satisfying the mono requirement, then for all* $\Gamma \vdash P, Q : X$*, if* $[\![P]\!] = [\![Q]\!]$ *then* $P \simeq Q$*.*

Corollary 2.19 (soundness)**.** *All well-typed closed ground returners reduce to a normal form. Explicitly: for all* $\vdash M : FG$ *there exists some* $\vdash V : G$ *such that:*

$$M \longrightarrow^{\star} \mathbf{return}\ V$$

Adequacy proofs

Using logical relations

- By Levy for CBPV
- Contributed for λ_{eff} (Hermida's lifting)
- Contributed for λ_{mon} ($\top\top$-lifting)

Levy (2004), Hermida (1993), Lindley, Stark (2005), Doczkal, Schwinghammer (2009)

Typed macro expressability

One concept can express another if there is a *local* translation function _ that:

- is homomorphic on the base calculus,
- replaces new syntactic constructs without rearranging the whole program,
- translates terms $\emptyset \vdash M : FG$ such that $M \longrightarrow^* \textbf{return}\ V \Longleftrightarrow$ M $^*\textbf{return}\ V$,

- translates terms $\emptyset \vdash M : X$ to terms $\emptyset \vdash \underline{M} : \underline{X}$.

Felleisen (1990)

λ_{eff} CAN MACRO EXPRESS λ_{mon}

$$\underline{\hat{\mu}^{\varepsilon}(N)}_{E} = \mathrm{op}^{\varepsilon}\ \{\underline{N}_{E}\}\ (\lambda x.\mathbf{return}\ x)$$

$$\underline{[\![N]\!]^{\varepsilon}}_{E} = \mathbf{handle}\ \underline{N}_{E}\ \mathbf{with}\ H^{E}_{\varepsilon}$$

$$\text{for } (\varepsilon \sim (N_u, N_b)) \in E \text{ and}$$

$$H^{E}_{\varepsilon} \coloneqq \left\{\mathbf{return}\ V \mapsto \underline{N_u}_{E} V,\ \mathrm{op}^{\varepsilon} p\, f \mapsto \underline{N_b}_{E}\, p\, f\right\}$$

$$\cup \left\{\mathrm{op}^{\varepsilon'} p\, f \mapsto \mathrm{op}^{\varepsilon'} p\, f \,\middle|\, \varepsilon \neq \varepsilon' \in E\right\}$$

$$\underline{\mathbf{leteffect}\ \varepsilon \succ e\ \mathbf{be}\ (\alpha.C, N_u, N_b)\ \mathbf{in}\ N}_{E} = \underline{N}_{E,\varepsilon\sim(N_u,N_b)}$$

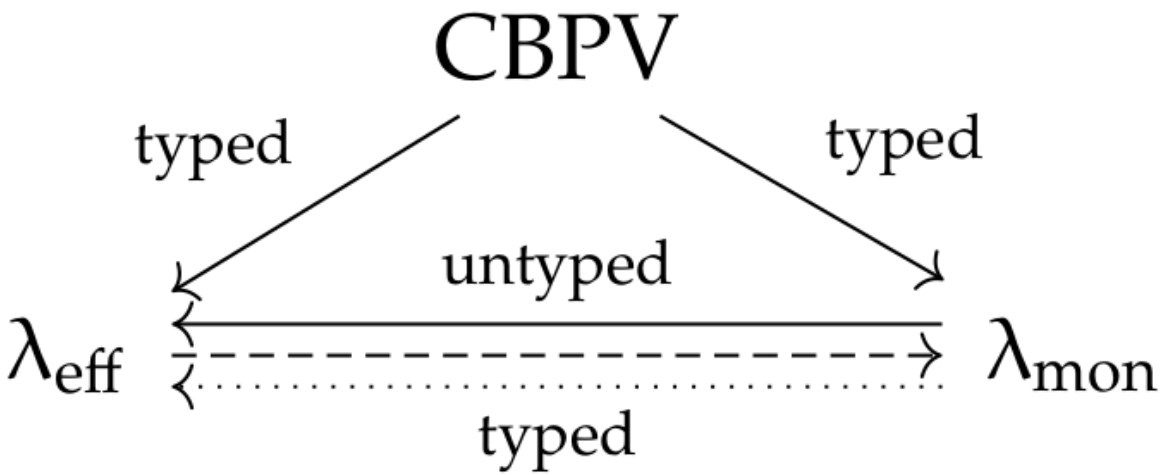
Focus in this thesis

Produce negative results: Prove that *no* translation exists with the help of denotational semantics

λ_{mon} CAN NOT TYPED MACRO EXPRESS λ_{eff}

- There are only finitely many terms for every type in λ_{mon}
- Some types in λ_{eff} have countably many observationally distinguishable terms
- Given a translation $\lambda_{\text{eff}} \to \lambda_{\text{mon}}$, take the type $\underline{F1}$
- $\underline{F1}$ has k terms
- $F1$ has more than k observationally distinguishable terms
- Derive a contradiction

The big picture



Contribution

- Adequacy proof for the set theoretic model for calculus of effect handlers λ_{eff}
- Adequate denotational semantics for calculus of monadic reflection λ_{mon}
- Definition of (typed) macro expressability
- Proof that λ_{mon} is macro expressible in λ_{eff}
- Proof that λ_{eff} is not macro typed expressible in λ_{mon}

Future work

- Show that λ_{mon} is not typed macro expressible in λ_{eff};
- extend the type system of λ_{eff} to typed macro express λ_{mon};
- do similar comparison for calculus of delimited control (partially solved).

Future work

- Show that λ_{mon} is not typed macro expressible in λ_{eff};
- extend the type system of λ_{eff} to typed macro express λ_{mon};
- do similar comparison for calculus of delimited control (partially solved).

- Formalise this in Coq?

Related work / Bibliography

- Paul Blain Levy. Call-By-Push-Value: A Functional/Imperative Synthesis, volume 2 of Semantics Structures in Computation. Springer, 2004.
- Ohad Kammar, Sam Lindley, and Nicolas Oury. Handlers in action. SIGPLAN Not. 48(9):145–158, September 2013.
- Andrzej Filinski. Monads in action. SIGPLAN Not., 45(1):483–494, January 2010.
- Matthias Felleisen. On the expressive power of programming languages. In Science of Computer Programming, pages 134–151. Springer-Verlag, 1990.

Value relations $\boxed{R_A \subseteq [\![A]\!] \times \lambda_{\mathbf{eff}}\, A}$

$$R_1 := \{\langle \star, V\rangle | V \simeq ()\}$$

$$R_{A_1 \times A_2} := \left\{\langle\langle a_1, a_2\rangle, V\rangle \middle| \exists V_1, V_2 \forall i. V_i : A_i.\, \langle a_i, V_i\rangle \in R_{A_i}, V \simeq (V_1, V_2)\right\} \qquad R_0 := \emptyset$$

$$R_{A_1 + A_2} = \bigcup_{i \in \{1,2\}} \left\{\langle \iota_i a, V\rangle \middle| \exists V' : A_i.\, \langle a, V'\rangle \in R_{A_i}, V \simeq \mathbf{inj}_i\, V'\right\}$$

$$R_{U_E C} := \{\langle a, V\rangle | \exists M : C.\, \langle a, M\rangle \in R_{E,C}, V \simeq \{M\}\}$$

Computation relations $\boxed{R_{E,C} \subseteq |[\![C]\!]| \times \lambda_{\mathbf{eff}\, E,C}}$

$$\begin{aligned}
R_{E,FA} &:= R'_{E,FA} \cup \{\langle \mathbf{return}\ a, M\rangle | \exists V : A.\, \langle a, V\rangle \in R_A, M \simeq \mathbf{return}\ V\} \\
R_{E,A\to C} &:= R'_{E,A\to C} \cup \{\langle f, M\rangle | \forall\, \langle a, V\rangle \in R_{E,A}, \langle f(a), M\ V\rangle \in R_{E,C}\} \\
R_{E,\top} &:= R'_{E,\top} \cup \{\langle \star, M\rangle | M \simeq \langle\rangle\} \\
R_{E,C_1 \& C_2} &:= R'_{E,C_1 \& C_2} \cup \left\{\langle\langle c_1, c_2\rangle, M\rangle \middle| \exists M_1 : C_1, M_2 : C_2.M \simeq \langle M_1, M_2\rangle, \forall i.\, \langle c_i, M_i\rangle \in R_{E,C_i}\right\}
\end{aligned}$$

where

$$\begin{aligned}
R'_{E,C} := \{ & \\
& \langle \mathrm{op}_a(\lambda b.c), N\rangle \mid \exists V M. \\
& \langle a, V\rangle \in R_{E,A} \quad (\mathrm{op} : A \to B) \in E \\
& \langle \lambda b.c, \lambda x.M\rangle \in R_{E,B\to C} \\
& N \simeq \mathrm{op}\ V\ (\lambda x.M) \\
\}&
\end{aligned}$$

Handler relations $\boxed{R_{A\,E\Rightarrow E'\,C} \subseteq [\![A\ {}^{E}\!\Rightarrow^{E'} C]\!] \times \mathbf{handlers}(A\ {}^{E}\!\Rightarrow^{E'} C)}$

$$\begin{aligned}
R_{A\,E\Rightarrow E'\,C} := \{ & \\
& \langle\langle D_\gamma, f_\gamma\rangle, H\rangle \mid \\
& (\mathrm{op}_i : A \to B) \in E, \quad D_\gamma = \langle |[\![C]\!]|, [\![-]\!](\gamma)\rangle, \\
& \forall i. \exists M N_i.\, \langle f_\gamma, \lambda x.M\rangle \in R_{E',A\to C} \wedge, \\
& \forall\, \langle a, V\rangle \in R_{E,A}, \langle k, \lambda x.M'\rangle \in RB \to C. \\
& \left\langle [\![\mathrm{op}_i]\!](\gamma)\ (\lambda x.kx \gg\!= f)\ a, N_i[V/p, (\lambda x.\mathbf{handle}\ M'\ \mathbf{with}\ H)/k]\right\rangle \\
\}&
\end{aligned}$$

Figure 3.8: λ_{eff}-calculus logical relations

Value relations $\boxed{R_{\Theta,\Sigma,A}(\rho) \subseteq [\![A]\!](\theta) \times \boldsymbol{\lambda}^{\Sigma}_{\mathbf{mon}}(A)}$

$$R_{\Theta,\Sigma,1}(\rho) := \{\langle \star, V\rangle | V \simeq ()\}$$

$$R_{\Theta,\Sigma,A_1\times A_2}(\rho) := \left\{\langle\langle a_1, a_2\rangle, V\rangle \middle| \exists V_1, V_2 \forall i. V_i : A_i.\, \langle a_i, V_i\rangle \in R_{\Sigma,A_i}(\rho), V \simeq (V_1, V_2)\right\}$$

$$R_{\Theta,\Sigma,0}(\rho) := \emptyset$$

$$R_{\Theta,\Sigma,A_1+A_2}(\rho) = \bigcup_{i=1,2} \left\{\langle \iota_i a, V\rangle \middle| \exists V' : A_i.\, \langle a, V'\rangle \in R_{\Theta,\Sigma,A_i}(\rho), V \simeq \mathbf{inj}_i\, V'\right\}$$

$$R_{\Theta,\Sigma,U_e C}(\rho) := \{\langle a, V\rangle | \exists M : C.\, \langle a, M\rangle \in R_{\Theta,\Sigma,e,C}(\rho), V \simeq \{M\}\} \qquad R_{\Theta,\Sigma,\alpha}(\rho) := \rho(\alpha)$$

Computation relations $\boxed{R^{v}_{\Theta,\Sigma,e,C}(\rho) \subseteq |[\![C]\!](\theta)| \times \boldsymbol{\lambda}^{\Sigma,e}_{\mathbf{mon}}(C)}$

$$R^{v}_{\Theta,\Sigma,\perp,FA}(\rho) := \{\langle a, \mathbf{return}\ V\rangle | \exists \langle a', V\rangle \in R_{\Sigma,A}(\rho), a = \mathbf{return}\ a'\}$$
$$R^{v}_{\Theta,\Sigma,\varepsilon,FA}(\rho) := \{\langle a, \mathbf{return}\ V\rangle | \exists \langle a', V\rangle \in R_{\Sigma,A}(\rho), a = \mathbf{return}\ a'\} \cup$$
$$\left\{\langle a, \widehat{\mu}^{\varepsilon}(N)\rangle \middle| (\varepsilon \sim \alpha.C) \in \Sigma, \langle a, N\rangle \in R_{\Theta,\Sigma,e,C[A/\alpha]}(\rho)\right\}$$

$$R^{v}_{\Theta,\Sigma,e,A\to C}(\rho) := \{\langle f, \lambda x.M\rangle | \forall \langle a, V\rangle \in R_{\Theta,\Sigma,e,A}(\rho), \langle f(a), (\lambda x.M)\ V\rangle \in R_{\Theta,\Sigma,e,C}(\rho)\}$$

$$R^{v}_{\Theta,\Sigma,e,\top}(\rho) := \{\langle \star, \langle\rangle\rangle\} \qquad R^{v}_{\Theta,\Sigma,e,C_1\&C_2}(\rho) := \left\{\langle\langle c_1, c_2\rangle, (M_1, M_2)\rangle \middle| \langle c_i, M_i\rangle \in R_{\Theta,\Sigma,e,C_i}(\rho)\right\}$$

$\top$- ***and*** $\top\top$***-liftings*** $\boxed{(R^{v}_{\Theta,\Sigma,e,C}(\rho))^{\top} \subseteq (|[\![C]\!](\theta)| \to |[\![FG]\!](\theta)|) \times \mathfrak{X}^{\Sigma,e}_{mon}(FG[C])}$ ***and***
$\boxed{(R^{v}_{\Theta,\Sigma,e,C}(\rho))^{\top\top} \subseteq |[\![C]\!](\theta)| \times \boldsymbol{\lambda}^{\Theta,\Sigma,e}_{\mathbf{mon}}(C)}$

$$(R^{v}_{\Theta,\Sigma,e,C}(\rho))^{\top} := \left\{\langle f, \mathcal{X}\rangle \middle| \emptyset[\Theta] \mid \emptyset[\emptyset] \vdash_{\Sigma[\Sigma],\perp[e]} \mathcal{X}[\ ] : FG[C]. \forall \langle a, M\rangle \in R^{v}_{\Theta,\Sigma,e,C}(\rho)\ .\ \exists V. \mathcal{X}[M] \simeq \mathbf{return}\ V \wedge \langle fa, \mathbf{return}\ V\rangle \in R^{v}_{\Sigma,\perp,FG}(\rho)\right\}$$

$$(R^{v}_{\Theta,\Sigma,e,C}(\rho))^{\top\top} := \left\{\langle c, M\rangle \middle| \forall \langle f, \mathcal{X}\rangle \in (R^{v}_{\Theta,\Sigma,e,C}(\rho))^{\top}. \exists V. \mathcal{X}[M] \simeq \mathbf{return}\ V \wedge \langle fc, \mathbf{return}\ V\rangle \in R^{v}_{\Sigma,\perp,FG}(\rho)\right\}$$

$$R_{\Theta,\Sigma,e,C}(\rho) := (R^{v}_{\Theta,\Sigma,e,C}(\rho))^{\top\top}$$

Figure 4.8: λ_{mon}-calculus logical relations

Lemma 2.17 (basic lemma). *For all* $\Gamma \vdash P : X$, $\gamma \in [\![\Gamma]\!]$ *and* $\langle V_x \rangle_{x \in \mathrm{Dom}(\Gamma)}$*:*

$$(\textit{for all } x \in \mathrm{Dom}\ (\Gamma)\textit{:}\ \langle \pi_x \gamma, V_x \rangle \in R_{\Gamma(x)}) \implies \left\langle [\![P]\!]\gamma, P[V_x/x]_{x \in \mathrm{Dom}(\Gamma)} \right\rangle \in R_X$$

Lemma 4.10 (basic lemma). *For all* Θ, Σ, *derivations* $\Theta \mid \Gamma \vdash_{\Sigma,e} P : X$, $\langle \emptyset \mid \Sigma \vdash_k A_\alpha : \mathbf{Val} \rangle_{\alpha\in\Theta}$, $\theta = \langle [\![A_\alpha]\!] (\star) \rangle_{\alpha\in\Theta}$, $\rho = \left\langle R_{\emptyset,\Sigma,A_\alpha}(\star) \right\rangle_{\alpha\in\Theta}$, $\gamma \in [\![\Gamma]\!](\theta)$, *and* $\langle V_x \rangle_{x\in \mathrm{Dom}(\Gamma)}$,

if for all $x \in \mathrm{Dom}\ (\Gamma)$: $\langle \pi_x \gamma, V_x \rangle \in R_{\Theta,\Sigma,e,\Gamma(x)}\ (\rho)$ *then* $\langle [\![P]\!](\theta)\gamma, P_{[V_x/x]_{x\in\mathrm{Dom}(\Gamma)}} \rangle \in R_{\Theta,\Sigma,e,X}\ (\rho)$

and for all Θ, Σ, $\langle \emptyset \mid \Sigma \vdash_k A_\alpha : \mathbf{Val} \rangle_{\alpha\in\Theta}$, $\theta = \langle [\![A_\alpha]\!] (\star) \rangle_{\alpha\in\Theta}$, $\rho = \left\langle R_{\emptyset,\Sigma,A_\alpha}(\star) \right\rangle_{\alpha\in\Theta}$, *and* $\emptyset \mid \Sigma \vdash_k A, B : \mathbf{Val}$, $\forall(\varepsilon \sim (\alpha.C, N_u, N_b)) \in \Sigma$

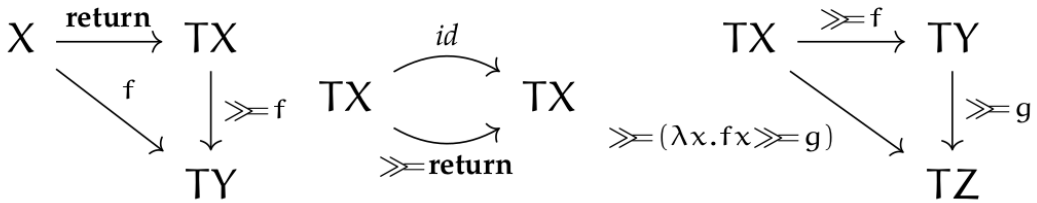
$$\langle [\![N_u]\!] (\theta)(\star), N_u \rangle \in R_{(\alpha,\Theta),\Sigma,e,\alpha\to C} \left(\rho \left[\alpha \mapsto R_{\emptyset,\Sigma,A}(\star) \right] \right)$$

and

$$\langle [\![N_b]\!] (\theta)(\star), N_b \rangle \in R_{(\alpha_1,\alpha_2,\Theta),\Sigma,e,U_e(C[\alpha_1/\alpha])\to U_e(\alpha_1\to C[\alpha_2/\alpha])\to C[\alpha_2/\alpha]} \left(\rho \left[\alpha_1 \mapsto R_{\emptyset,\Sigma,A}(\star), \alpha_2 \mapsto R_{\emptyset,\Sigma,B}(\star) \right] \right)$$

Monad

Definition 2.1. *A monad is a triple* $\langle \mathsf{T}, \mathbf{return}, \gg\!= \rangle$ *where* T *is a class function* **Set** $\to$ **Set** *and* $\mathbf{return}^{\mathsf{X}} : \mathsf{X} \to \mathsf{TX}$ *and* $\gg\!=^{\mathsf{X},\mathsf{Y}} : \mathsf{TX} \times (\mathsf{X} \to \mathsf{TY}) \to \mathsf{TY}$ *are families of functions such that for every* $\mathsf{f} : \mathsf{X} \to \mathsf{TY}$, $\mathsf{g} : \mathsf{Y} \to \mathsf{TZ}$ *the following diagrams commute:*



Algebra for a monad T

A *T-algebra* for a monad T is a pair $C = \langle |C|, c \rangle$ where $|C|$ is a set and $c : T\,|C| \to |C|$ is a function satisfying:

$$c(\mathbf{return}\ x) = x \qquad\qquad c(\mathbf{fmap}\ c\ xs) = c(xs >>= \mathrm{id})$$

for all $x \in |C|$ and $xs \in T^2\,|C|$. $|C|$ is called the *carrier* and we call c the *algebra structure*.