A Completeness proof for intuitionistic epistemic logic

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August 28, 2020

We prove soundness and completeness for IEL^- in a classical setting. The general proof idea and structure is taken from Artemov and Protopopescu (2016), the Lindenbaum lemma's formalization is based on previous works by Wehr (2018) (constructive extension of contexts) and Bentzen (2019) (reasoning about delta chain in a proof assistant). Wehr proves completeness for classical first-order predicate logic. Bentzen proves completeness for (non-intuitonistic) propositional modal logic S_5 in the Lean proof assistant using classical reasoning. The proofs have been formalized in the Coq proof assistant. We outline how the proofs can be adapted for IEL .

IEL and IEL^- were introduced by Artemov and Protopopescu (2016) as logics which model an intuitionistic conception of knowledge. Formulas in intuitionistic epistemic logic are generated by the following Backus-Naur-Form:

$$\phi, \psi := p_i \mid \neg\psi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \rightarrow \psi \mid \mathsf{K}\,\phi$$

where p_i is any proposition from a set of (possibly countably infinite) propositional variables. The set of all formulas is denoted by $\mathcal{F}$. In the accompanying Coq development, variables are represented by natural numbers.

The provability relation for IEL^- can be presented as a natural deduction calculus whose rules are listed in Figure 1. We use the common notation $\Gamma \vdash \phi$ to express that from a set of formulae Γ, the formula ϕ can be derived. One obtains a natural deduction calculus for IEL by adding intuitionistic reflection that is $\Gamma \vdash \mathsf{K}\,\phi \rightarrow \Gamma \vdash \neg\neg\phi$.

Figure 1: Natural deduction rules for IEL^-

$$\text{A}\ \frac{\varphi \in \Gamma}{\Gamma \vdash \varphi} \qquad \text{E}\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \qquad \text{II}\ \frac{\Gamma \cup \{\varphi\} \vdash \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash (\varphi \rightarrow \psi)}$$

$$\text{IE}\ \frac{\Gamma \vdash (\varphi \rightarrow \psi) \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi} \qquad \text{KImp}\ \frac{A \vdash \mathsf{K}\,(\varphi \rightarrow t)}{A \vdash \mathsf{K}\,\varphi \rightarrow \mathsf{K}\,t} \qquad \text{IntRefl}\ \frac{A \vdash s}{A \vdash \mathsf{K}\,s}$$

$$\text{DIL}\ \frac{A \vdash s}{A \vdash s \vee t} \qquad \text{DIR}\ \frac{A \vdash t}{A \vdash s \vee t} \qquad \text{DE}\ \frac{A, s \vdash \psi \quad A, t \vdash \psi \quad A \vdash s \vee t}{A \vdash q}$$

$$\text{CI}\ \frac{A \vdash s \quad A \vdash t}{A \vdash s \wedge t} \qquad \text{CEL}\ \frac{A \vdash s \wedge t}{A \vdash s} \qquad \text{CER}\ \frac{A \vdash s \wedge t}{A \vdash t}$$

We call any set of formulas Γ a context. In our development, contexts are represented as predicates $\Gamma : \mathcal{F} \rightarrow \mathbb{P}$. With the intended reading, that $\phi \in \Gamma$ if and only if $\Gamma\,\phi$ is provable.

We define an adjunction operation on contexts which adds a formula into a context. This operations is defined as $\Gamma, \phi := \lambda z. \Gamma\, z \vee (z = \phi)$. Two contexts are extensionally equivalent, denoted by $A \equiv B$ if $\forall \phi \in \mathcal{F} : A\, \phi \iff B\, \phi$. Extensionally equivalent contexts derive the same formulae.

Lemma 1 (Weakening)**.** *For contexts Γ, Ω if $\Gamma \subseteq \Omega$ and $\Gamma \vdash \psi$ then $\Omega \vdash \psi$.*

Proof. The proof is by induction on the derivation $\Gamma \vdash \psi$. □

Our representation of contexts as functions allows for infinite contexts, which will ease proving completeness. However, reasoning along the lines " Since $\Gamma \vdash \phi$, there are $\gamma_1, \ldots, \gamma_{n-1}$ s.t. $\gamma_1, \ldots, \gamma_{n-1} \vdash \phi$" becomes more complicated. We therefore introduce the concept of a finite context and proof that $\Gamma \vdash \phi$ if and only if there is a finite subcontext proving ϕ. In Coq we use the notion of listability for expressing finiteness, that is a context is finite iff there is a list containing every formula from the context.

Lemma 2 (Finite Derivation Lemma)**.** *For any context Γ and formula ϕ, $\Gamma \vdash \phi$ if and only if there exists a finite context $\gamma \subseteq \Gamma$ with $\gamma \vdash \phi$.*

Proof. The " $\Longrightarrow$ "-direction is proven by induction on the derivation. The only-if direction is proven by weakening. □

Lindenbaum Lemma

The Lindenbaum lemma states that any context not deriving a formula ψ can be extended to a theory, that is a set such that $\phi \in \Gamma \iff \Gamma \vdash \phi$, which also does not derive ψ and adding any formula, not already contained, would allow deriving ψ.

In this section we consider a fixed context Γ and a formula ψ not derivable in Γ, i.e. $\Gamma \nvdash \psi$. The construction works by greedily extending the context. We assume an enumeration $\delta : \mathbb{N} \to \mathcal{F}$ which is surjective, i.e. for every formula $\phi \in \mathcal{F}$ there exists (at least) an n s.t. $\delta\, n = \phi$.

We begin by defining a function, which extends a context by a formula, if this does not lead to the context deriving ψ.

$$\Gamma + \phi := \begin{cases} \Gamma, \phi & \text{if } \Gamma, \phi \nvdash \psi \\ \Gamma & \text{otherwise} \end{cases}$$

We realize this function not as an if-statement (since we have not proven decidability for $\vdash$) but instead encode the case distinction as a disjunction:

$$\Gamma + \phi := \lambda \gamma . \Gamma\, \gamma \vee (\Gamma, \phi \nvdash \psi \wedge \gamma = \phi)$$

Lemma 3. *Either $\Gamma + \phi \equiv \Gamma$ and $\Gamma + \phi \vdash \psi$ or $\Gamma + \phi \equiv \Gamma, \phi$.*

Proof. Either $\Gamma, \phi \vdash \psi$ or $\Gamma, \phi \nvdash \phi$ by classical reasoning. After doing a case analysis on that, the proof is easy. □

The construction used to generate the maximal set is to define a chain of sets $\Delta_0 = \Gamma \subseteq \Delta_1 \subseteq \cdots \subseteq \Delta_m \subseteq \ldots$. Where Δ_{i+1} is obtained by extending Δ_i by the i-th formula in our enumeration, if this does not cause the set to derive ψ.

$$\Delta_i := \begin{cases} \Gamma & \text{if i = 0} \\ (\Delta_{i-1} + \phi_{i-1} & \text{otherwise} \end{cases}$$

Lemma 4 (Lindenbaum)**.** *If* $\Gamma \nvdash \psi$*, then* $\Delta := \bigcup_{i \in \mathbb{N}} \Delta_i$

- *is a theory, that is* $\Delta \vdash \phi \iff \phi \in \Delta$
- *does not derive* ψ*, formally* $\Delta \nvdash \psi$
- *is prime*[1]*, i.e.* $\Gamma \vdash s \vee t \rightarrow \Gamma \vdash s \vee \Gamma \vdash t$
- *is a superset of* Γ*, i.e.* $\Gamma \subseteq \Delta$

[1] Note, that for theories an equivalent definition would be $(s \vee t) \in \Gamma \rightarrow s \in \Gamma \vee t \in \Gamma$.

We will prove each of the above parts independently. We start with the easiest:

Lemma 5. $\Gamma \subseteq \Delta$

Proof. Since $\Delta_0 = \Gamma$ and $\Delta_0 \subseteq \Delta$ the inclusion is true. □

To proof that Δ does not derive ψ we first proof the following lemma:

Lemma 6. *For all* i*,* $\Delta_i \nvdash \psi$*.*

Proof. The proof is by induction on i. The $i = 0$ case is easy, in the successor case, we can establish the property using Lemma 3. □

Proving that the union of all Δ_i's does not derive ψ is a bit harder. Our strategy is the following one: We proof that $\Delta \vdash \psi \iff \exists i, \Delta_i \vdash \psi$.

Lemma 7 (Chain lemma)**.** $\Delta \vdash \psi \iff \exists i, \Delta_i \vdash \psi$

Proof. Showing the only-if part is easy: Assume $\Delta_i \vdash \phi$. Since $\Delta_i \subseteq \Delta$ weakening completes the proof.

For the if-part, we use the fact that $\Delta \vdash \phi$ implies that there is a finite context $\Delta_\phi \subseteq \Delta$ s.t. $\Delta_\phi \subseteq \Delta$ and $\Delta_\phi \vdash \phi$. Since for every element γ in Δ there is a minimal i s.t. $\gamma \in \Delta_i$ there is an index I (the maximum of the individual minimal I's) s.t. all formulas in γ are Δ_I. Thus (by weakening) $\Delta_I \vdash \phi$. □

Lemma 8. $\Delta \nvdash \phi$*.*

Proof. Assume $\Delta \vdash \phi$, then by the Chain lemma there is an i s.t. $\Delta_i \vdash \phi$. This contradicts Lemma 6. □

Thus we have proven that Δ does not derive φ.

Next we will prove that Δ is a theory.

Lemma 9 (Theory Lemma)**.** *For all* $\phi \in \mathcal{F} : \Delta \vdash \phi \iff \phi \in \Delta$*.*

Proof Sketch. Assume $\Delta \vdash \phi$. We want to prove $\phi \in \Delta$. Since δ enumerates $\mathcal{F}$, there exists an i s.t. $\phi_i = \phi$. We do a case distinction on wether ϕ was inserted into Δ_{i+1} or not (formally a case distinction on Lemma 3).

- If ϕ was not inserted, $\Delta_i, \phi \vdash \psi$. Thus $\Delta \vdash \phi \to \psi$, thus we can prove $\Delta \vdash \psi$ by the implication elimination rule, which contradicts lemma 8.

- If ϕ was inserted, it trivially also is in Δ, since $\Delta_i \subseteq \Delta$.

  The converse direction is proven using the assumption rule. □

For establishing the primeness property, we need an additional lemma.

Lemma 10. *If* $\Delta \nvdash \phi$ *then* $\Delta \vdash \phi \to \psi$.

Proof. The proof is similiar to Lemma 9. We do a case distinction on wether ϕ was inserted or not. If it was inserted, we arrive at a contradiction, since it would obviously be derivable.

If ϕ was not inserted there is a j such that $\Delta_j, \phi \vdash \psi$; by implication introduction we obtain $\Delta_j \vdash \phi \to \psi$ and by weakening we proof the result. □

With this Lemma in place, we can show primeness:

Lemma 11. Δ *is prime, i.e.* $\Delta \vdash (s \vee t) \implies \Delta \vdash s \vee \Delta \vdash t$

Proof. By classical reasoning $\Delta \vdash s \vee \Delta \nvdash s$ and $\Delta \vdash t \vee \Delta \nvdash t$. We get 4 cases, all cases were s or t are derivable from Δ are straightforward. The case $\Delta \nvdash s, \Delta \nvdash t$ is most interesting as we need to arrive at a contradiction. We know that Δ is consistent and does not derive ψ (since $\Gamma \nvdash \psi$). By the lemma above we get both $\Delta \vdash s \to \psi$ and $\Delta \vdash t \to \psi$. Applying the disjunction elimination rule we obtain $\Delta \vdash \phi$ and we have arrived at a contradiction. □

Kripke Models

Definition 1 (IEL-Models)**.** *An IEL*$^-$ *model is a quadruple* $(\mathcal{W}, \mathcal{V}, \leq, \leq_v)$ *where*

- $\mathcal{W}$ *is a type, whose elements represent the possible worlds*

- $\mathcal{V} : \mathcal{W} \to \mathbb{N} \to \mathbb{P}$ *is the valuation function, which maps worlds and propositional variables to propositions.*

- $\leq$ *is a preorder on the worlds, the cognition relation*

- $\leq_v$ *is the verification relation on worlds*

To be a valid IEL$^-$ *models the following constraints have to be satisified:*

- *The verification relation has to be a subrelation of the cognition relation, i.e.* $u \leq_v v \implies u \leq v$ *for all* $u, v \in \mathcal{W}$.

- *If* $u \leq v$ *and* $v \leq_v w$ *then* $u \leq_v w$ *for all* $u, v \in \mathcal{W}$.

IEL models have the additional condition, that every world has a $\leq_v$*-successor , i.e.* $\forall w \in \mathcal{W} : \exists w' \in \mathcal{W} : w \leq_v w'$.

The distinction between IEL$^-$ and IEL models will be made by defining a predicate on the models.

Next we have to define semantic entailment in the model. That is we define a semantic entailment relation between formulas and worlds. This entailment relation is represented as a function $\Vdash$ from models, a formula and a world in the model to propositions. We use the common notation $\mathcal{M}, w \Vdash \phi$ to denote that ϕ is semantically entailed in the model $\mathcal{M}$ at world w .

Definition 2. *We can define semantic entailment by induction on the formula.*

- $\mathcal{M}, w \Vdash p_i := \mathcal{V}_w(i)$
- $\mathcal{M}, w \Vdash \phi \wedge \psi := (\mathcal{M}, w \vDash \phi) \wedge (\mathcal{M}, w \vDash \psi)$
- $\mathcal{M}, w \Vdash \phi \vee \psi := (\mathcal{M}, w \vDash \phi) \vee (\mathcal{M}, w \vDash \psi)$
- $\mathcal{M}, w \Vdash \phi \to \psi := \forall w', w \leq w' \implies (\mathcal{M}, w' \vDash \phi) \implies (\mathcal{M}, w' \vDash \psi)$
- $\mathcal{M}, w \Vdash \mathsf{K}\,\phi := \forall w \leq_v w' : w' \vDash \phi$
- $\mathcal{M}, w \Vdash \bot := \bot$

Note, that the last case establishes, that $\bot$ can never be true at a model (unless we have proven falsity).

Completeness

In order to prove **completeness** we define a canonical model $\mathcal{M}_c$.

Definition 3 (Canonical Model)**.** *The canonical model is a quadruple* $(W, V, \prec, \prec_v)$ *with*

- $W := \{\Gamma | \Gamma$ *is a consistent prime theory*$\}$
- $\mathcal{V}(s, \Gamma) := s \in \Gamma$
- $\Gamma \prec \Gamma' :\Leftrightarrow \Gamma \subseteq \Gamma'$
- $\Gamma \prec_v \Gamma' :\Leftrightarrow \Gamma_{\mathbf{K}} \subseteq \Gamma'$ *where* $\Gamma_{\mathbf{K}} := \{\phi | \mathsf{K}\,\phi \in \Gamma\}$. [2]

[2] Note that (we will also proof this result below) for every theory Γ, $\Gamma_{\mathbf{K}} \subseteq \Gamma' \implies \Gamma \subseteq \Gamma'$, since in both IEL and IEL$^-$ intuitionistic reflection holds.

Lemma 12. *The canonical model is a model for both IEL$^-$ and IEL .*

Proof. We need to check that

- $\prec$ is a preorder
- $\prec_v \subseteq \prec$
- $\Gamma \prec \Omega \wedge \Omega \prec_v \Lambda \to \Gamma \prec_v \Lambda$

The first one is simple and left out. For the second assume $\Gamma \prec_v \Omega$. We need to show that $\Gamma \prec \Omega$, i.e. $\Gamma \subseteq \Omega$. Let $\phi \in \Gamma$, by co-reflection (and the fact that Γ is a theory), we get $\mathsf{K}\,\phi \in \Gamma$. Thus $\phi \in \Gamma_{\mathbf{K}}$ and since $\Gamma_{\mathbf{K}} \subseteq \Omega$ the claim $\phi \in \Omega$ follows.

To prove the third one, first note that $\Gamma \subseteq \Omega \implies \Gamma_{\mathbf{K}} \subseteq \Omega_{\mathbf{K}}$.[3] We can

[3] The proof is simple: Let $\phi \in \Gamma_{\mathbf{K}}$ thus by definition $\mathsf{K}\,\phi \in \Gamma$. Since $\Gamma \subseteq \Omega$, $\mathsf{K}\,\phi \in \Omega$ immediately follows and by definition $\phi \in \Omega_{\mathbf{K}}$.

assume $\Gamma \prec \Omega$ that is $\Omega \subseteq \Gamma$, thus $\Gamma_{\mathbf{K}} \subseteq \Omega_{\mathbf{K}}$ by our insight and $\Gamma \prec_v \Lambda$ that is $\Omega_{\mathbf{K}} \subseteq \Lambda$. With transitivity of $\subseteq$ we obtain the desired result.

For IEL , we addditionally have to prove, that there are no worlds without a $\prec_v$ successor. To main idea to prove, that a world Γ has a $\prec_v$ successor, is to apply the Lindenbaum-Lemma to $\Gamma_{\mathbf{K}}$ avoiding to derive $\bot$. To proof that this extension does not derive $\bot$ (and therefore is consistent and thus a world in the canonical model), it suffices to prove $\Gamma_{\mathbf{K}} \nvdash \bot$, which can be done by using reasoning similar to the Artemov Lemma below. [4] □

[4] Assume $\Gamma_{\mathbf{K}} \vdash \bot$ then by the Artemov Lemma $\Gamma \vdash \mathsf{K}\bot$. Using, that IEL derives $\mathsf{K}\bot \to \bot$, we could proof $\Gamma \vdash \bot$. This would contradict our assumption, that Γ is consistent (since Γ is a world in the model).

The canonical model has the property, that a formula is semantically entailed at a world if and only if it is an element of the respective world. We will need a small additional lemma first.

Lemma 13 (Artemov Lemma)**.** *For any context* $\Gamma, \Gamma_{\mathbf{K}} \vdash \phi \to \Gamma \vdash \mathsf{K}\,\phi$.

Proof. Assume $\Gamma_{\mathbb{K}} \vdash \phi$. By the finite derivation lemma, this implies the existence of finitely many γ_i such that $\{\gamma_1, \ldots, \gamma_m\} \vdash \phi$ with $\gamma_1, \ldots, \gamma_m \in \Gamma_{\mathbb{K}}$. We can shift the context into the formula (we iteratively apply the equivalence $s, \Gamma \vdash t \iff \Gamma \vdash s \to t$) and thus have a proof of $\vdash \gamma_1 \to \gamma_2 \to \cdots \to \gamma_m \to \phi$. By using positive introspection and iteratively applying the `KIMP` rule, that is $\mathsf{K}\,s \to t \vdash \mathsf{K}\,s \to \mathsf{K}\,t$, we obtain $\vdash \mathsf{K}\,\gamma_1 \to \cdots \to \mathsf{K}\,\gamma_m \to \mathsf{K}\,\phi$. We can now "unshift" the $\mathsf{K}\,\gamma_i$ into the context and get $\{\mathsf{K}\,\gamma_1, \ldots, \mathsf{K}\,\gamma_m\} \vdash \mathsf{K}\,\phi$. But now $\Gamma \vdash \mathsf{K}\,\phi$ since $\mathsf{K}\,\gamma_1, \ldots, \mathsf{K}\,\gamma_m \in \Gamma$ since $\gamma_1, \ldots, \gamma_m$ are contained in $\Gamma_{\mathbf{K}}$. □

In the proof of the next lemma we will use the contrapositive of the Artemov Lemma.

Lemma 14 (Truth Lemma)**.** *For all* $\phi \in \mathcal{F}, \Gamma \in \mathcal{W}(\mathcal{M}_c) : \mathcal{M}_c, \Gamma \vDash \phi \iff \phi \in \Gamma$.

Proof. Induction on the formula ϕ with the worlds quantified. We write $\Gamma \Vdash \phi$ and leave out the model $\mathcal{M}_c$.

Case $\phi = \bot$: In the if-part we get falsity $\Gamma \Vdash \bot = \bot$ as an assumption. Since we can proof anything from falsity, the proof is done. In the only-if part, we know that $\Gamma\ \bot$ is provable. Since our models are consistent theories, meaning $\neg\Gamma\ \bot$, we have a proof of falsity.

Case $\phi = s \wedge t$: We get the induction hypotheses $\forall\Gamma : \Gamma \Vdash s \iff s \in \Gamma$ and $\forall\Gamma : \Gamma \Vdash t \iff t \in \Gamma$. For the if-part, we can assume $\Gamma \Vdash (s \wedge t)$ thus $\Gamma \Vdash s$ and $\Gamma \Vdash t$ by the definition of $\Vdash$. Therefore we have a proof of $s \in \Gamma$ and $t \in \Gamma$. We can now apply the introduction rule for conjunction and obtain a proof of $\Gamma \vdash (s \wedge t)$, and since Γ is a theory, this proofs $s \wedge t \in \Gamma$. For the other direction, assume $(s \wedge t) \in \Gamma$, since Γ is a theory $\Gamma \vdash s \wedge t$ and by conjunction elimination $\Gamma \vdash s$ and $\Gamma \vdash t$. Again using that Γ is a theory and applying the inductive hypothesis concludes the proof.

Case $\phi = s \vee t$: The proof is (somewhat) similar to the conjunction proof, as we get the same inductive hypotheses. It can be done by equational reasoning and needs that in a prime theory Γ, $s \vee t \in \Gamma \iff s \in \Gamma \vee t \in \Gamma$:

$$\begin{aligned}
\Gamma \Vdash s \vee t &\iff s \vee t \in \Gamma & \\
\Gamma \Vdash s \vee \Gamma \Vdash t &\iff s \vee t \in \Gamma & \text{(Definition of } \Vdash\text{)} \\
\Gamma \Vdash s \vee - \Vdash t &\iff s \in \Gamma \vee t \in \Gamma & (\Gamma \text{ is a prime theorie)} \\
s \in \Gamma \vee t \in \Gamma &\iff s \in \Gamma \vee t \in \Gamma & \text{(inductive hypothesis)}
\end{aligned}$$

Since the last equivalence is a tautology (and we only used rewriting with equivalences), the proof is done.

Case $\phi = \mathsf{K}\,\psi$: For the only-if-part we can assume $\mathsf{K}\,\psi \in w$ and need to show $w \Vdash \mathsf{K}\,\psi$ that is for all $w' \succ_v w : w' \Vdash \psi$. Let w' be such a world. Since $w \prec_v w'$ and $\mathsf{K}\,\phi \in w$, we know $\phi \in w'$, by the inductive hypothesis we deduce $w' \Vdash \phi$.

We show the if-part by contraposition. So assume $\mathsf{K}\,\phi \notin \Gamma$. We have to show $\Gamma \nVdash \mathsf{K}\,\phi$. Since Γ is a theory, $\Gamma \nvdash \mathsf{K}\,\phi$, thus $\Gamma_{\mathbf{K}} \nvdash \phi$ by the Artemov Lemma. Using the Lindenbaum-Lemma we can extend $\Gamma_{\mathbf{K}}$ to a consistent prime theory Γ' without ϕ i.e. $\phi \notin \Gamma'$. But by the inductive hypothesis, this implies $\Gamma' \nVdash \phi$. We have $\Gamma \prec_v \Gamma'$, since Γ' is an extension of $\Gamma_{\mathbf{K}}$ but $\Gamma' \nVdash \phi$, therefore $\Gamma \nVdash \mathsf{K}\,\phi$.

Case $\phi = s \to t$: We will start by proving the only-if part, since it is easier. So assume $s \to t \in \Gamma$. We need to prove that for every world $\Gamma' \succ \Gamma$ if $\Gamma' \Vdash s$ then $\Gamma' \Vdash t$. So let Γ' be such a world; therefore assume $\Gamma \prec \Gamma'$ and $\Gamma' \Vdash s$. We need to prove $\Gamma' \Vdash t$. By the inductive hypothesis and since Γ' is a theory, it suffices to show $\Gamma' \vdash t$. We apply the elimination rule for implications and have to show $\Gamma' \vdash (s \to t)$ and $\Gamma' \vdash s$. For the second part, it suffices to show $s \in \Gamma'$ (by the inductive hypothesis), but that was our assumption. For the second, we know $(s \to t) \in \Gamma$. Using $\Gamma \prec \Gamma'$ we get $(s \to t) \in \Gamma'$.

For the if-part, we show the contraposition. That is we assume $(s \to t) \notin \Gamma$ and need to prove $\Gamma \nVdash s \to t$. For this it suffices to show the existence of a $\Gamma' \prec \Gamma$, in which s is a semantic consequence and t is not, i.e. $\Gamma' \Vdash s$ and $\Gamma' \nVdash t$. We construct such a Γ' as the Lindenbaum-extension of s, Γ, avoiding to derive t. That such a extension is a world our model is straightforward, since the Lindenbaum-Lemma generates consistent maximal theories.[5] The Lindenbaum-Lemma also guarantees that such an extension does not derive t if the initial context does not derive it. That s can be derived is straightforward. Of course we need to check the precondition, that is $s, \Gamma \nvdash t$. Assume $s, \Gamma \vdash t$, by implication introduction we get $\Gamma \vdash s \to t$ which would contradict our assumption. Therefore $\Gamma \nvdash s \to t$ and we can apply the Lindenbaum-Lemma.

[5] Notice, that at such a Γ', s is always semantically entailed due to monotonicity.

□

Theorem 1 (Completeness)**.** *If a formula is entailed in every model it is derivable from an empty context.*

Proof. We show the contraposition. Assume $\nvdash \phi$. By the Lindenbaum Lemma there is a prime Δ such that $\phi \notin \Delta$. By the Truth Lemma, in the canonical model $\Delta \nVdash \phi$. But then the formula is not entailed in every model. □

Conclusion

We proved soundness and completeness in a classical setting and mechanized the proof for IEL^- in Coq. Next steps could include

- mechanize the proof for IEL
- proof decidability of IEL or/and IEL^-
- try to proof completeness constructively

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