► DOMINIK KIRST AND IAN SHILLITO, *A succinct and verified completeness proof for first-order bi-intuitionistic logic.*

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In the 1970s, Cecylia Rauszer provided the foundations for bi-intuitionistic logic, an extension of intuitionistic logic with a binary operator $\prec$ called exclusion, dual to the intuitionistic implication $\to$. Her work spanned over most approaches to non-classical logics, ranging from algebras [11, 15], Kripke semantics [13, 14, 17], sequent calculus [12], to Hilbert systems [11, 12]. The impressiveness and exhaustiveness of Rauszer's study of bi-intuitionistic logic is not only measured by the variety of fields she introduced bi-intuitionistic in, but by the analysis in each case of *both* the propositional and first-order logic.

Unfortunately, through time several mistakes were detected in Rauszer's work. First, her sequent calculus for propositional bi-intuitionistic logic was shown by Pinto and Uustalu [10] not to admit cut, contradicting her claim [12, Result 2.3]. To correct this, they provided a calculus based on sequents with richer structure, which they proved to admit cut. Secondly, a confusion around the status of the deduction theorem led Goré and Shillito [4] to notice the conflation in Rauszer's work of two *different* propositional bi-intuitionistic logics. This conflation resulted in an incorrect completeness proof for the propositional case, ultimately resolved by Goré and Shillito. Finally, the errors contained in the propositional case continue being present in Rauszer's work on the first-order case as noted by Shillito [19], who failed to fix the proof in this setting. So, to date, no completeness proof for first-order bi-intuitionistic logic (FOBIL) along the lines of Rauszer's argument is known.

To our knowledge, the only other candidate proof was given by Klemke in 1971 [6], thereby in fact predating Rauszer's work. He attributes the semantics of the logic to Grzegorczyk [5] and uses a Henkin-style argument to construct a universal model. However, its correctness is questioned by Olkhovikov and Badia [8], who write:

> "Incidentally, there is an alternative completeness argument by Klemke, where bi-intuitionistic predicate logic is studied possibly for the first time in print (and, as far as we know, independently from Rauszer's work) and that contains other errors."

As his proof strategy is technically involved and, being written in fairly old style (and German language), the presentation is rather inaccessible to a broader audience, it is hard to assess whether these errors are locally fixable or as substantially unfixable as Rauszer's.

We therefore opt for an alternative route to settle the completeness of FOBIL once and for all: we present a *succinct* proof based on standard techniques, coming in a modern (and English) presentation for easy assessment, and use the Coq proof assistant to *verify* our argument, therefore leaving no room for ambiguity and error.

In that vein, our formal investigation finally establishes solid foundations for FOBIL, and simultaneously tightly connects the provability of the *constant domain axiom* in this logic with constant domain models. That is, contrarily to the propositional case, first-order bi-intuitionistic logic is known not to be a conservative extension of first-order intuitionistic logic [18, p.56][7, 19]: it derives the constant domain axiom (CD), displayed below, which is not provable in the purely intuitionistic counterpart [2].

$$\text{(CD)} \qquad \forall x(\varphi(x) \vee \psi) \to (\forall x \varphi(x) \vee \psi)$$

Here, the variable x is required not to occur freely in ψ. As the name suggests, this axiom characterises the *constant domain property* on models in the Kripke semantics for the intuitionistic language [5, 2, 9]. Rauszer suggested that this connection between the axiom and the property on models should also hold in the bi-intuitionistic setting [13, 18]. The first-order Kripke semantics she developed uses frames for intuitionistic logic satisfying the constant domain property, thus capturing the semantics for FOCDIL, i.e. first-order intuitionistic logic extended with the (CD) axiom. Our results provide a confirmation of Rauszer's suggestion by showing FOBIL complete relative to the constant domain semantics, notably settling the logic as a conservative extension of FOCDIL [18, p.57][1].

In fact, our completeness proof for bi-intuitionistic logic mostly follows the textbook proof of Gabbay, Shehtman, and Skvortsov [3] for FOCDIL. As our only actually novel idea, we observe that their use of a custom Lindenbaum lemma exploiting the (CD) axiom to obtain successor worlds in a universal model can be dualised, namely, to obtain also predecessor worlds, exploiting a dualisation of the (CD) axiom:.

$$\text{(DCD)} \qquad (\exists x \varphi(x) \wedge \psi) \prec \exists x(\varphi(x) \wedge \psi)$$

While (CD) is used as a theorem, i.e. $\top \vdash$ (CD), we exploit the contradictory nature of (DCD) in our custom lemma, i.e. (DCD) $\vdash \bot$. The remaining argument is also streamlined to dispose of the usual Henkin-style syntax extensions to obtain a particularly succinct presentation that is feasible to verify in Coq with little technical overhead.

In summary, the contributions of our work are as follows:

- We give a succinct completeness proof for FOBIL based on standard techniques, closing a gap in the literature not featuring a single unquestionably correct proof.
- We illustrate the tight connection of FOBIL and FOCDIL, in that our completeness proof of the former extends and dualises the one of the latter.
- We provide a Coq mechanisation verifying all definitions and results in our work for absolute clarity and correctness.
- As a by-product, we give the, to the best of our knowledge, first mechanisation of the completeness of FOCDIL and the conservativity of FOBIL over FOCDIL.

We shall here just give a brief overview of our completeness proof. The overall strategy is a standard Kripke canonical model construction, namely a syntactic model M^c over the domain of terms and the worlds W^c being Henkin prime theories, i.e. consistent and deductively closed sets of formulas that are well-behaved regarding quantifiers and disjunctions. Certain sets can be extended into worlds of that model as follows:

Lemma 1 (Lindenbaum Lemmas). *Let Γ and Δ be sets of formulas.*

1. *For closed Γ and Δ such that $\Gamma \nvdash \Delta$, there is a Henkin prime theory $\Gamma' \supseteq \Gamma$ such that $\Gamma' \nvdash \Delta$.*
2. *For any Henkin theory Γ and formulas ψ_1, ψ_2 such that $\Gamma, \psi_1 \nvdash \psi_2$, there is a Henkin prime theory $\Gamma' \supseteq \Gamma$ with $\psi_1 \in \Gamma'$ and $\psi_2 \notin \Gamma'$.*
3. *For any Henkin prime theory Γ and formulas ψ_1, ψ_2 with $\psi_1 \nvdash \overline{\Gamma}, \psi_2$, there is a Henkin prime theory $\Gamma' \subseteq \Gamma$ with $\psi_1 \in \Gamma'$ and $\psi_2 \notin \Gamma'$.*

Here, (1) is the usual iterative construction following an enumeration of formulas, while additionally (2) exploits (CD) and (3) exploits (DCD). The variants (2) and (3) are needed for the implication and exclusion case of the truth lemma, respectively:

Lemma 2 (Truth lemma). *For every $\Gamma \in W^c$ we have $\psi \in \Gamma$ iff $\Gamma \Vdash \psi$.*

To conclude completeness for $\Gamma \models \varphi$, variant (1) is used to extend a supposed non-derivability $\Gamma \nvdash \varphi$ into a world of of M^c, then conflicting $\Gamma \models \varphi$ via the truth lemma.

Theorem 3 (Completeness). *If $\Gamma \cup \{\varphi\}$ is closed and $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.*

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