Post's Problem in Constructive Mathematics

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Abstract

We study a solution to Post's problem, i.e. the existence of a semi-decidable but undecidable Turing degree strictly below the halting problem, from the perspective of constructive mathematics. This perspective allows to combine two approaches:

First, using a synthetic approach to computability à la Richman and Bauer, we assume axioms that identify the function space $\mathbb{N} \to \mathbb{N}$ with computable functions and allow a simple definition of Turing reductions based on sequentially continuous functionals. Such axioms are incompatible with strong choice axioms, but remain consistent even in the presence of classical axioms such as the law of excluded middle (LEM) in suitable foundations such as the Calculus of Inductive Constructions, a variant of constructive type theory.

Secondly, we approximate the logical strength of the result by showing that assuming the limited principle of omniscience (LPO), a weak fragment of LEM, is enough to construct a solution to Post's problem. This suggests a future project in the spirit of constructive reverse mathematics, namely to analyse whether LPO is in turn implied by the assumption of a solution to Post's problem, and therefore necessary in our proof.

Post's Problem Posed by Emil Post in 1944 [14], Post's problem asks whether there are semi-decidable but undecidable predicates that are not Turing-reducible from the halting problem. Post's problem has been a crucial open question driving research in computability theory until a breakthrough came with the positive solution by Friedberg and Muchnik [8, 12] in 1956/57. They introduced independently what is now known as the priority method, in order to show that there exist two semi-decidable, Turing-reduction incomparable degrees. Lerman and Soare's solution to Post's problem [11] constructs a so-called low simple predicate directly, rather than proving the full Friedberg-Muchnik theorem constructing two incomparable predicates.

Contribution The first three authors have recently presented their synthetic Coq mechanisation of a low simple predicate using LEM for Σ_2 formulas [18]. Σ_2-LEM is strictly stronger than LPO, which is equivalent to Σ_1-LEM [1]. Combining this result with an observation by the fourth author [13], working in a non-mechanised and non-synthetic yet constructive setting, we contribute a mechanised synthetic proof that already LPO induces a solution of Post's problem.

Synthetic Oracle Computability We briefly summarise the synthetic framework for oracle computability developed by Forster, Kirst, and Mück [6, 7] based on work of Bauer [2, 3] and related to Swan's development in HoTT [17].

Technically, we work in the Calculus of Inductive Constructions, representing sets as predicates of type $\mathbb{N} \to \mathbb{P}$ and their characteristic relations of type $\mathbb{N} \to \mathbb{B} \to \mathbb{P}$, but all definitions can be reproduced in any other constructive foundation.

A functional $F : (\mathbb{N} \to \mathbb{B} \to \mathbb{P}) \to \mathbb{N} \to \mathbb{B} \to \mathbb{P}$ is considered oracle-computable if there is an underlying computation tree $\tau : \mathbb{N} \to \mathbb{B}^* \rightharpoonup \mathbb{N} + \mathbb{B}$ capturing the extensional behaviour of F by

$$\forall Rxb.\ FRxb \ \leftrightarrow\ \exists qs\ as.\ \tau x\,;R \vdash qs\,;as \land \tau\ x\ as \triangleright \mathsf{out}\ b$$

where the *interrogation relation* $\sigma; R \vdash qs; as$ is inductively defined for $\sigma : \mathbb{B}^* \rightharpoonup \mathbb{N} + \mathbb{B}$ as

$$\frac{}{\sigma\,;R \vdash [\,]\,;[\,]} \qquad\qquad \frac{\sigma\,;R \vdash qs\,;as \qquad \sigma\,as \triangleright \mathsf{ask}\ q \qquad Rqa}{\sigma\,;R \vdash qs \mathbin{+\!\!+} [q]\,;as \mathbin{+\!\!+} [a]}$$

and where we write $\mathsf{ask}\ q$ and $\mathsf{out}\ o$ for the respective injections into the sum type $\mathbb{N} + \mathbb{B}$.

Computation trees provide a notion of sequential continuity that singles out the functionals operating like oracle machines. The general definition yields a notion of Turing reductions $P \preceq_T Q$, by requiring an oracle computation F that maps Q to P, and a notion of relative semi-decidability $\mathcal{S}_Q(P)$, by requiring an oracle computation F that maps Q to the positive part of P. The connection of the two notions is given as expected:

Lemma 1 (Theorem 35 in [6])**.** *If $\mathcal{S}_Q(P)$ and $\mathcal{S}_Q(\overline{P})$ then $P \preceq_T Q$.*

Moreover, as part of Post's theorem, relative semi-decidability relates to logical complexity:

Lemma 2 (Theorem 43 in [7])**.** *Assuming Σ_n-LEM, if P is Σ_{n+1} and Q is Σ_n, then $\mathcal{S}_Q(P)$.*

Here, Σ_n formulas are defined as $px := \exists x_1 : \mathbb{N}.\forall x_2 : \mathbb{N}.\exists x_3 : \mathbb{N} \ldots f(x, x_1, x_2, x_3, \ldots) = \mathsf{true}$ for a function $f : \mathbb{N}^{n+1} \to \mathbb{B}$. Given the synthetic setting, no computability requirement is needed for f.

While these lemmas are provable without any axioms for synthetic computability, to continue we assume an enumeration of semi-deciders, which is a variant of Richman's synthetic form of the standard axiom Church's thesis (CT) [15, 10]. This yields at the same time an enumeration W_e of the semi-decidable predicates and an enumeration Φ_e of oracle computations. We can then define the halting problem H by $H\,x := W_x\,x$ and the Turing jump P' of a predicatae P by $P'\,x := \Phi_x^P(x)\downarrow$ as well as show the former undecidable and the latter irreducible to P.

The Priority Method The priority method can be used to construct semi-decidable predicates S satisfying infinite sequences of positive requirements P_e and negative requirements N_e. Here, we consider the simplest form of the priority method, the finite injury priority method, as originally developed by Friedberg and Muchnik to solve Post's problem. We define S as the union of finite, cumulative stages, where L is the n-th stage if $n \rightsquigarrow L$ holds. The construction is parametric in a predicate $\gamma : \mathbb{N}^* \to \mathbb{N} \to \mathbb{N} \to \mathbb{P}$, used to determine whether an element can enter S at stage n.

$$\frac{}{0 \rightsquigarrow [\,]} \qquad \frac{n \rightsquigarrow L \quad \gamma_n^L\ x}{n+1 \rightsquigarrow x :: L} \qquad \frac{n \rightsquigarrow L \quad \forall x.\ \neg\, \gamma_n^L\ x}{n+1 \rightsquigarrow L}$$

We instantiate γ suitably to a function such that the following requirements are met by S:

$$P_e := W_e \text{ is infinite } \to W_e \cap S \neq \emptyset \qquad N_e := (\exists^\infty n.\ \Phi_e^S(e)[n]\downarrow) \to \Phi_e^S(e)\downarrow$$

Low Simple Predicates Since a synthetic notion of simple predicates has been defined by Forster and Jahn [5], we here focus on the aspect of lowness. A predicate P is low if its Turing jump P' is reducible to the halting problem H, i.e. if $P' \preceq_T H$. Note that, as desired, lowness of P rules out a reduction $H \preceq_T P$, as then $P' \preceq_T P$ would follow.

As a tool to establish reductions to H, limit computability was introduced by Shoenfield [16] and re-discovered by Gold [9]. Synthetically, we call a predicate $p : X \to \mathbb{P}$ limit-computable if there exists a function $f : X \to \mathbb{N} \to \mathbb{B}$ with

$$px \leftrightarrow \exists n.\forall m > n.\ f(x, m) = \mathsf{true} \quad \wedge \quad \neg px \leftrightarrow \exists n.\forall m > n.\ f(x, m) = \mathsf{false}.$$

Lemma 3 (Limit Lemma)**.** *Assuming LPO, if P is limit computable, then $P \preceq_T H$.*

Proof. If P is limit computable, then immediately by definition both P and $\overline{P}$ are Σ_2. Moreover, since the halting problem H is Σ_1, Lemma 2 together with LPO yields both $\mathcal{S}_H(P)$ and $\mathcal{S}_H(\overline{P})$. From there we conclude $P \preceq_T H$ with Lemma 1. □

As a result, we just need to prove that S' is limit-computable to establish lowness of S. We leave the details for the talk, but remark that LPO is used for the limit lemma as well as to verify that S fulfills the requirements N_e and that S' is limit computable. We conclude with the final theorem:

Theorem 1 (Post's Problem)**.** *Assuming LPO, a low simple predicate exists.*

Our proof also sheds light on an analytic setting. If any use of the enumerability axiom is replaced by an explicit construction in a model of computation, (or an informal use of the Church Turing thesis) it follows that a variant of LPO defined using Turing-computable $f : \mathbb{N} \to \mathbb{B}$, discussed in [4], is enough to otherwise constructively prove the existence of low simple predicates.

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