The Blurred Drinker Paradox

Constructive Reverse Mathematics of the Löwenheim-Skolem Theorem

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Background

Some work I do with Ian

Constructive completeness proofs for bi-intuitionistic logic:

- Propositional case:
  - Unrestricted completeness is equivalent to LEM ($\forall p : \mathbb{P}.\, p \vee \neg p$)
  - In specific constellations, Markov's Principle (MP) is enough: $\forall f : \mathbb{N} \to \mathbb{B}.\, \neg\neg(\exists n.\, f\ n = \text{true}) \to \exists n.\, f\ n = \text{true}$
  - In other constellations, other (non-comparable) principles play a role

- First-order case:
  - So far we know that unrestricted completeness is equivalent to LEM
  - No understanding of the restricted cases yet

Connects to the relation of FOL-Completeness to principles like the boolean prime ideal theorem, the ultrafilter lemma, weak König's lemma, the fan theorem etc.

Constructive Reverse Mathematics[1]

Classical reverse mathematics studies classically detectable equivalences?

- Which theorems are equivalent to the axiom of choice or weak König's lemma?
- Which theorems are equivalent to the continuum hypothesis?
- ...

Classical reverse mathematics studies constructively detectable equivalences?

- Which theorems are equivalent to LEM or MP?
- Which theorems are equivalent to which specific formulation of the axiom of choice?
- ...

Characterises the computational content of analysed theorems!

[1] Ishihara (2006); Diener (2018)

The Downwards Löwenheim-Skolem Theorem (DLS)

Definition (Elementary Submodels)

Given models $\mathcal{M}$ and $\mathcal{N}$, we call $h : \mathcal{M} \to \mathcal{N}$ an elementary embedding if

$$\forall \rho \varphi.\, \mathcal{M} \vDash_\rho \varphi \leftrightarrow \mathcal{N} \vDash_{h \circ \rho} \varphi.$$

If such an h exists, we call $\mathcal{M}$ an elementary submodel of $\mathcal{N}$.

Theorem (DLS)

Every model has a countable elementary submodel.

What is the computational content of the DLS theorem?

Classical Reverse Mathematics of DLS

$$\begin{aligned} \mathsf{DC}_A &:= \forall R : A \to A \to \mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N} \to A. \forall n.\, R\,(f\ n)\,(f\,(n+1)) \\ \mathsf{CC}_A &:= \forall R : \mathbb{N} \to A \to \mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N} \to A. \forall n.\, R\,n\,(f\ n) \end{aligned}$$

Theorem

The DLS theorem is equivalent to DC.

Sketch.

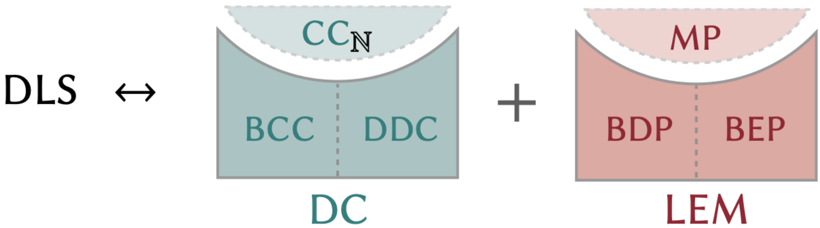
- To prove DLS from DC, arrange the iterative construction such that a single application of DC yields a path through all possible extensions that induces the resulting submodel.
- Starting with a total relation $R : A \to A \to \mathbb{P}$, consider (A, R) a model. Applying DLS, obtain an elementary submodel $(\mathbb{N}, R')$ so in particular R' is still total. Apply $\mathsf{CC}_{\mathbb{N}}$ to obtain a choice function for R' that is reflected back to A as a path through R.

□

Constructive Reverse Mathematics of DLS?

Does the DLS theorem still follow from DC alone or is there some contribution of LEM?

Does the DLS theorem still imply DC or is there some contribution of CC?



Classical Argument

The Drinker Paradox

In every bar, one can identify a person such that,
if they drink, then the whole bar drinks

$$\begin{aligned} \mathrm{DP}_A &:= \forall P : A \to \mathbb{P}.\, \exists x.\, P\,x \to \forall y.\, P\,y \\ \mathrm{EP}_A &:= \forall P : A \to \mathbb{P}.\, \exists x.\, (\exists y.\, P\,y) \to P\,x \end{aligned}$$

Fact

DP and EP are equivalent to LEM.

Proof.

To derive LEM from DP, given $p : \mathbb{P}$ use DP for $A := \{b : \mathbb{B} \mid b = \mathsf{false} \lor (p \lor \neg p)\}$ and $P : A \to \mathbb{P}$ defined by $P\,(\mathsf{true}, \_) := \neg p$ and $P\,(\mathsf{false}, \_) := \top$. $\square$

DLS using Henkin Environments

Definition (Henkin Environment)

Given a model $\mathcal{M}$, we call $\rho : \mathbb{N} \to \mathcal{M}$ a Henkin environment if for all φ:

$$\exists n.\, \mathcal{M} \vDash_\rho \varphi[\rho\, n] \;\to\; \mathcal{M} \vDash_\rho \dot{\forall}\varphi$$
$$\exists n.\, \mathcal{M} \vDash_\rho \dot{\exists}\varphi \;\to\; \mathcal{M} \vDash_\rho \varphi[\rho\, n]$$

Lemma

Every model with a Henkin environment has a countable elementary submodel.

Proof.

Given a model $\mathcal{M}$ and a Henkin environment ρ, we obtain a countable elementary submodel as the syntactic model $\mathcal{N}$ constructed over the domain $\mathbb{T}$ of terms by setting

$$f^{\mathcal{N}}\, \vec{t} \;:=\; f\, \vec{t} \qquad \text{and} \qquad P^{\mathcal{N}}\, \vec{t} \;:=\; P^{M}\, (\hat{\rho}\, \vec{t}). \quad \square$$

DLS assuming DC and LEM

Theorem

Assuming DC and LEM, the DLS theorem holds.

Proof.

Construct a Henkin environment in three steps:

1. Given some environment ρ, we know by DP and EP that, relative to ρ, Henkin witnesses for all formulas exist.
2. Applying CC we can simultaneously choose from these witnesses at once and therefore extend to some environment ρ'.
3. This describes a total relation on environments, through which DC yields a path that can be merged into a single environment, and that then must be Henkin. $\square$

Reverse Analysis

Theorem

Assuming $\mathrm{CC}_{\mathbb{N}}$*, the DLS theorem implies DC.*

Proof.

Following the outline from the beginning, using the assumption of $CC_{\mathbb{N}}$ to obtain a choice function in the countable elementary submodel. $\square$

So over $\mathrm{CC}_{\mathbb{N}}$ and LEM, the DLS theorem is equivalent to DC.

Refining the Use of LEM

The Blurred Drinker Paradox (BDP)

In every bar, there is an at most countable group such that,
if all of them drink, the the whole bar drinks

$$
\begin{aligned}
\mathrm{BDP}_A &:= \forall P : A \to \mathbb{P}. \exists f : \mathbb{N} \to A. (\forall y. P (f\, y)) \to \forall x. P\, x \\
\mathrm{BEP}_A &:= \forall P : A \to \mathbb{P}. \exists f : \mathbb{N} \to A. (\exists x. P\, x) \to \exists y. P (f\, y)
\end{aligned}
$$

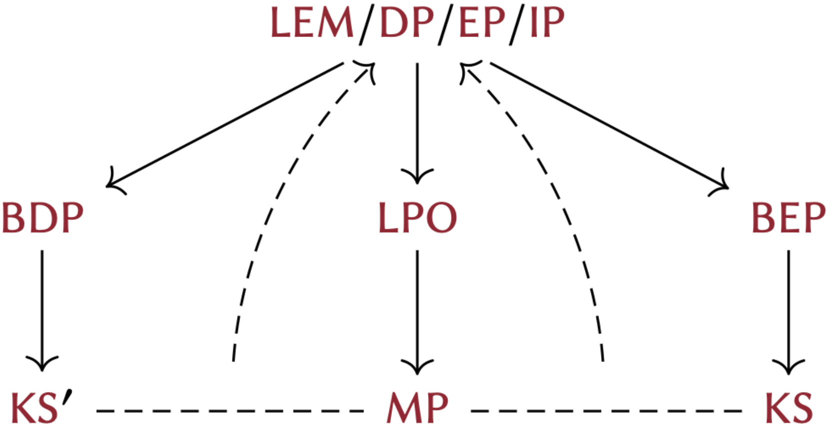
Fact

LEM decomposes into $\mathrm{BDP} + \mathrm{DP}_{\mathbb{N}}$ *and even* $\mathrm{BDP} + \mathrm{MP}$, *similarly for* BEP.

Proof.

The first decomposition is trivial. The latter follows since BDP implies Kripke's schema (KS) which is known to imply LEM in connection to MP. □

Classification of BDP



DLS using Blurred Henkin Environments

Definition (Henkin Environment)

Given a model $\mathcal{M}$, we call $\rho : \mathbb{N} \to \mathcal{M}$ a blurred Henkin environment if or all φ:

$$(\forall n.\, \mathcal{M} \vDash_\rho \varphi[\rho\, n]) \;\to\; \mathcal{M} \vDash_\rho \dot{\forall}\varphi$$

$$\mathcal{M} \vDash_\rho \dot{\exists}\varphi \;\to\; (\exists n.\, \mathcal{M} \vDash_\rho \varphi[\rho\, n])$$

Lemma

Every model with a blurred Henkin environment has a countable elementary submodel.

Proof.

Given a model $\mathcal{M}$ and a blurred Henkin environment ρ, we obtain a countable elementary submodel as the same syntactic model $\mathcal{N}$ constructed over the domain $\mathbb{T}$ from before. □

DLS assuming DC and BDP

Theorem

Assuming DC and BDP/BEP, the DLS theorem holds.

Proof.

Construct a blurred Henkin environment in three steps:

1. Given some environment ρ, we know by BDP/BEP that, relative to ρ, blurred Henkin witnesses for all formulas exist.
2. Applying CC we can simultaneously choose from these witnesses at once and therefore extend to some environment ρ'.
3. This describes a total relation on environments through which DC yields a path, that can be merged into a single environment, and that then must be blurred Henkin. $\square$

Reverse Analysis

Theorem

The DLS theorem implies BDP and BEP.

Proof.

Using the same pattern as in the previous analysis, basically DLS reduces BDP to the trivially provable $\mathrm{BDP}_{\mathbb{N}}$, respectively BEP to the trivially provable $\mathrm{BEP}_{\mathbb{N}}$. $\square$

So over $\mathrm{CC}_{\mathbb{N}}$, the DLS theorem decomposes into DC+BDP+BEP.

Refining the Use of DC

Blurred Choice Axioms

$$\begin{aligned}
\mathsf{BCC}_A &:= \forall R : \mathbb{N}{\to}A{\to}\mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N}{\to}A. \forall n. \exists m.\, R\, n\, (f\ m) \\
\mathsf{DDC}_A &:= \forall R : A{\to}A{\to}\mathbb{P}.\, \mathsf{dir}(R) \to \exists f : \mathbb{N}{\to}A.\, \mathsf{dir}(R \circ f)
\end{aligned}$$

Lemma

CC *decomposes into* BCC + $\mathsf{CC}_\mathbb{N}$ *and* DC *decomposes into* DDC + CC.

$$\mathsf{BDC}^2_A := \forall R : A^2{\to}A{\to}\mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N}{\to}A.\, \mathsf{tot}(R \circ f)$$

Lemma

BDC^2 *decomposed into* BCC + DDC.

Classification of Blurred Choice Axioms

$$
\begin{tikzcd}
\mathrm{DC} \arrow[rr] \arrow[d] & & \mathrm{CC} \arrow[d] \\
\mathrm{BDC}^2 \arrow[r] \arrow[d] & \mathrm{BDC} \arrow[r] & \mathrm{BCC} \\
\mathrm{DDC} & &
\end{tikzcd}
$$

DLS assuming BDC and BDP

Theorem

Assuming BDC^2 *and BDP/BEP, the DLS theorem holds.*

Proof.

Construct a blurred Henkin environment in three steps:

1. Given some environment ρ, we know by BDP/BEP that, relative to ρ, blurred Henkin witnesses for all formulas exist.
2. Applying BCC we can simultaneously choose from these witnesses at once and therefore extend to some environment ρ'.
3. This describes a directed relation on environments, through which DDC yields a path that can be merged into a single environment, and that then must be blurred Henkin. □

Reverse Analysis

Theorem

The DLS theorem implies BDC^2 *and therefore also* BCC *and* DDC.

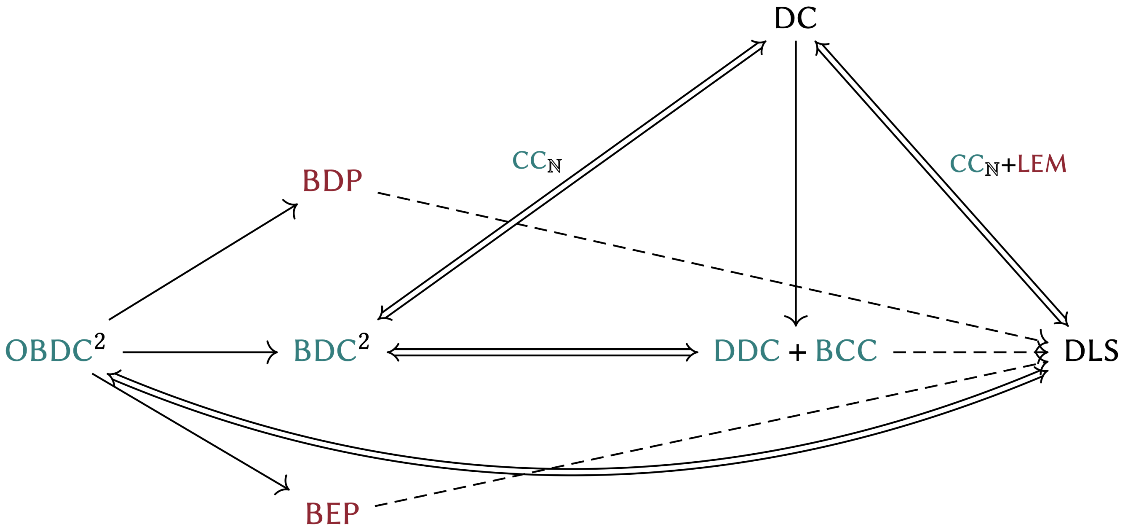
Proof.

Using the same pattern as in the previous analyses. $\square$

So the DLS theorem decomposes into $\mathsf{BDC}^2 + \mathsf{BDP} + \mathsf{BEP}$.

Conclusion

Overview



Remaining Questions?

- What happens with uncountable cardinalities?
  - Weaker forms of blurred drinker paradoxes, stronger forms of blurred choice principles
- Are the blurred principles weaker than the original?
  - We expect BDP $\not\to$ LEM, BCC $\not\to$ CC, and DDC $\not\to$ BCC
- What is the constructive status of the upwards Löwenheim-Skolem theorem?
  - Usual proof strategy uses compactness which is as non-constructive as completeness

Other things we could chat about while I'm here

- Synthetic Computability Theory
  - Undecidability, Oracle Computability, Post's Problem
- Computational Metamathematics
  - Incompleteness, Tennenbaum's theorem, Löb's Theorem
- Foundations of Mathematics
  - Comparison of Constructive Type Theory and Axiomatic Set Theory
- Interactive Theorem Proving
  - Everything is mechanised in the Coq proof assistant

Bibliography I

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