Separating Markov's Principles

An Introduction to Constructive Reverse Mathematics

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ABSTRACT

Markov's principle (MP) is an axiom in some varieties of constructive mathematics, stating that Σ_1^0 propositions (i.e. existential quantification over a decidable predicate on $\mathbb{N}$) are stable under double negation. However, there are various non-equivalent definitions of decidable predicates and thus Σ_1^0 in constructive foundations, leading to non-equivalent Markov's principles. While this fact is well-reported in the literature, it is often overlooked, leading to wrong claims in standard references and published papers.

In this paper, we clarify the status of three natural variants of MP in constructive mathematics, by giving respective equivalence proofs to different formulations of Post's theorem, to stability of termination of computations, to completeness of various proof

1 INTRODUCTION

Markov's Principle (MP) is a central principle in constructive mathematics, nowadays most commonly stated as follows [7, 8, 46]:

$$\forall f : \mathbb{N} \to \mathbb{B}.\ \neg\neg(\exists n.\ f n = \mathsf{true}) \to \exists n.\ f n = \mathsf{true}$$

It states that Σ_1^0 propositions, i.e. existential quantifications over decidable predicates, are stable under double negation. While not generally accepted in all flavours of constructive mathematics, it is a principle of the Russian school led by Markov [3, Ch. 3]. It also has a central status in constructive reverse mathematics [8, 20], where it is well-known to be equivalent to a multitude of principles spanning many areas of mathematics and theoretical computer science, going back to Gödel's insight that his completeness proof for first-order logic w.r.t. Tarski semantics requires MP.

Outline

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- Which theorems are equivalent to the axiom of choice or similar principles?
- Which theorems are equivalent to which comprehension principles?
- Many more, see Friedman (1976) and Simpson (2009)

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Constructive reverse mathematics studies constructively detectable equivalences:

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- Which theorems are equivalent to which specific formulation of the axiom of choice?
- Many more, see Ishihara (2006) and Diener (2018)

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- Many more, see Ishihara (2006) and Diener (2018)

Characterises the computational content of analysed theorems

Some Typical Principles

Some Typical Principles

Fragments of the excluded middle:

$$
\begin{aligned}
\mathsf{LEM} &:= \forall P : \mathbb{P}.\, P \lor \neg P \\
\mathsf{LPO} &:= \forall f : \mathbb{N}\to\mathbb{B}.\, (\exists n.\, f\ n = \mathsf{true}) \lor (\forall n.\, f\ n = \mathsf{false}) \\
\mathsf{MP} &:= \forall f : \mathbb{N}\to\mathbb{B}.\, \neg\neg(\exists n.\, f\ n = \mathsf{true}) \to \exists n.\, f\ n = \mathsf{true}
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Fragments of the axioms of choice:

$$\begin{aligned}
\mathsf{AC} &:= \forall AB. \forall R : A \to B \to \mathbb{P}.\, \mathsf{tot}(R) \to \exists f : A \to B. \forall x.\, R\, x\, (f\ x) \\
\mathsf{DC} &:= \forall A.\, \mathsf{inhab}(A) \to \forall R : A \to A \to \mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N} \to A. \forall n.\, R\, (f\ n)\, (f\ (n+1)) \\
\mathsf{CC} &:= \forall A. \forall R : \mathbb{N} \to A \to \mathbb{P}.\, \mathsf{tot}(R) \to \exists f : \mathbb{N} \to A. \forall n.\, R\, n\, (f\ n)
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To unveil fine distinctions, we use CIC as a modest base system

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Often very subtle, we use Coq to systematically track dependencies

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Classical foundations are usually blind for LEM and $CC_{\mathbb{N}}$...

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Theorem (Boolos, Burgess, Jeffrey (2002))

The downward Löwenheim-Skolem theorem is equivalent to DC.

- Only "$DC - CC_{\mathbb{N}}$" is necessary but also "LEM − MP" (K. and Zeng 2024)

Outline

1. Constructive reverse mathematics
2. Markov's principle and its equivalents
3. Separating Markov's principles (informally)
4. Separating Markov's principles (formally)
5. Ongoing work

Base System: Constructive Type Theory

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Features not included in CIC:

- Classical axioms that allow case distinctions of the form $P \lor \neg P$
- Choice principles turning total relations $R : X \to Y \to \mathbb{P}$ into functions $f : X \to Y$

Type-Theoretic Notions of Existence

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Usual definitions in the prominent constructive type theories:

$$
\begin{aligned}
\text{In CIC:} &\quad \exists x.\, p\, x \\
\text{In HoTT:} &\quad \|\Sigma x.\, p\, x\| \\
\text{In MLTT:} &\quad \Sigma x.\, p\, x
\end{aligned}
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Ascendingly stronger induced choice principles:

$$\Delta_1\text{-AC in CIC:} \quad \forall f : \mathbb{N} \to \mathbb{B}.\, (\exists n.\, f\ n = \text{true}) \to \Sigma n.\, f\ n = \text{true}$$
$$\text{UC in HoTT:} \quad \forall AB.\forall R : A{\to}B{\to}\mathbb{P}.\, \text{tot}(R) \to \text{fun}(R) \to \exists f : A{\to}B.\forall x.\, R\,x\,(f\ x)$$
$$\text{AC in MLTT:} \quad \forall AB.\forall R : A{\to}B{\to}\mathbb{P}.\, \text{tot}(R) \to \exists f : A{\to}B.\forall x.\, R\,x\,(f\ x)$$

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The alternative $\neg\neg\Sigma x.\, p\, x$ blocks any choice principles but trivialises MP...

Stating Markov's Principles

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$$\begin{aligned}
\mathsf{MP}_{\mathbb{P}} &:= \forall p : \mathbb{N} \to \mathbb{P}.\, (\forall n.\, p\,n \lor \neg p\,n) \to \neg\neg(\exists n.\, p\,n) \to \exists n.\, p\,n \\
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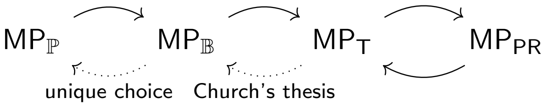

Confusion about Markov's Principles

Lots of false statements on Wikipedia and nLab

Confusion about Markov's Principles

Lots of false statements on Wikipedia and nLab, also (anonymous) experts can be wrong:

"$\mathrm{MP}_{\mathbb{B}}$ *states that a Turing machine that does not loop necessarily terminates*"

Restating Markov's Principles

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Different underlying notions of decidability of predicates $p : \mathbb{N} \to \mathbb{P}$:

$$
\begin{aligned}
\mathbb{P}\text{–}\mathcal{D}(p) &:= \forall n.\, p\,n \vee \neg p\,n \\
\mathbb{B}\text{–}\mathcal{D}(p) &:= \exists f : \mathbb{N} \to \mathbb{B}. \qquad \forall x.\, p\,x \leftrightarrow f\,x = \text{true} \\
\mathsf{T}\text{–}\mathcal{D}(p) &:= \exists f : \mathbb{N} \to \mathbb{B}.\, \text{computable} \to \quad \forall x.\, p\,x \leftrightarrow f\,x = \text{true} \\
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Restating Markov's Principles

Different underlying notions of decidability of predicates $p : \mathbb{N} \to \mathbb{P}$:

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\mathbb{P}{-}\mathcal{D}(p) &:= \forall n.\, p\, n \lor \neg p\, n \\
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\mathsf{T}{-}\mathcal{D}(p) &:= \exists f : \mathbb{N} \to \mathbb{B}.\, \text{computable} \to \forall x.\, p\, x \leftrightarrow f\, x = \text{true} \\
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Induced notions of $C{-}\Sigma_1$ propositions/predicates relative to $C{-}\mathcal{D}(p)$:

$$
\begin{aligned}
\mathsf{MP}_{\mathbb{P}} &\leftrightarrow \forall P : \mathbb{P}.\, \mathbb{P}{-}\Sigma_1(P) \to \neg\neg P \to P \\
\mathsf{MP}_{\mathbb{B}} &\leftrightarrow \forall P : \mathbb{P}.\, \mathbb{B}{-}\Sigma_1(P) \to \neg\neg P \to P \\
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Post's Theorem (cf. Forster, K., Smolka (2019))

Theorem

Assuming MP_C, if $C{-}\Sigma_1(p)$ and $C{-}\Sigma_1(\overline{p})$ then $C{-}\mathcal{D}(p)$.

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Theorem

Assuming MP_C*, if* $C{-}\Sigma_1(p)$ *and* $C{-}\Sigma_1(\overline{p})$ *then* $C{-}\mathcal{D}(p)$.

Proof.

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Theorem

Assuming MP_C, if $C\text{−}\Sigma_1(p)$ and $C\text{−}\Sigma_1(\overline{p})$ then $C\text{−}\mathcal{D}(p)$.

Proof.

First show $\mathbb{P}\text{−}\mathcal{D}(p)$ by applying MP_C to the constructively provable $\forall n.\, \neg\neg(p\,n \lor \neg p\,n)$.

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Then observe that $\mathbb{P}{-}\mathcal{D}(p)$ together with $C{-}\Sigma_1(p)$ and $C{-}\Sigma_1(\overline{p})$ implies $C{-}\mathcal{D}(p)$. □

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The converse holds.

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Proof.

Assume $C{-}\Sigma_1(P)$ and $\neg\neg P$. For $p\,n := P$ observe that $C{-}\Sigma_1(p)$ and $C{-}\Sigma_1(\overline{p})$.

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Assume $C{-}\Sigma_1(P)$ and $\neg\neg P$. For $p\,n := P$ observe that $C{-}\Sigma_1(p)$ and $C{-}\Sigma_1(\overline{p})$.
By the assumption of Post's theorem, obtain $C{-}\mathcal{D}(p)$ and conclude P. □

Completeness Theorems (cf. Forster, K., Wehr (2021))

Theorem

Assuming MP_C*, if* $T \vDash \varphi$ *for a* $C{-}\Sigma_1$ *theory, then* $T \vdash \varphi$*.*

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Assuming MP_C, if $T \vDash \varphi$ for a $C{-}\Sigma_1$ theory, then $T \vdash \varphi$.

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Observe that $T \vdash \varphi$ is $C{-}\Sigma_1$, so assuming MP_C and $T \vDash \varphi$ it is enough to show $\neg\neg(T \vdash \varphi)$.

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Assuming MP_C, if $T \vDash \varphi$ for a $C{-}\Sigma_1$ theory, then $T \vdash \varphi$.

Proof.

Observe that $T \vdash \varphi$ is $C{-}\Sigma_1$, so assuming MP_C and $T \vDash \varphi$ it is enough to show $\neg\neg(T \vdash \varphi)$. So on assumption of $T \nvdash \varphi$, construct a model $M \vDash T$ with $M \vdash \varphi$, contradicting $T \vDash \varphi$. □

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The converse holds.

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Assume $C{-}\Sigma_1(P)$ and $\neg\neg P$. For $T\,\psi := P$ and $\varphi := \bot$, observe that T is $C{-}\Sigma_1$ and $T \vDash \bot$.

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- Every C-extended natural number $N : \mathbb{N} \to \mathbb{P}$ that is not infinite is actually finite, where N is a C-extended natural number if $C{-}\mathcal{D}(N)$ and N is monotone, and finitude means $\exists n.\, N\, n$.

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- Every C-tree $\tau : \mathbb{B}^* \to \mathbb{P}$ that is not infinite is finite, where τ is a C-tree if $C{-}\mathcal{D}(N)$, $\tau\epsilon$, and $\tau\,l$ whenever $\tau\,l l'$, finitude means $\exists n. \forall l. |l| \leq n \to \neg\tau\,l$.

Outline

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2. Use Brouwer's free choice sequences to directly refute MP
   - Observed in another paper by Kreisel (1958b)
   - Also scales to constructive type theory (Coquand and Mannaa, 2016)
   - If done carefully, yields separation proof (Smorynski, 1973)

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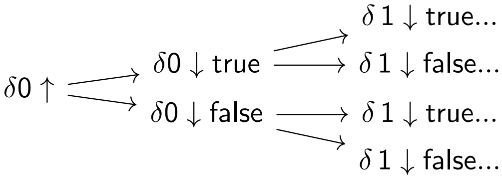

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$$\delta 0 \uparrow \;\longrightarrow\; \delta 0 \downarrow \text{true} \;\longrightarrow\; \delta\, 1 \downarrow \text{true}\ldots \quad / \quad \delta\, 1 \downarrow \text{false}\ldots$$

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Gives rise to an interpretation of intuitionistic logic

Invalidating Markov's Principle

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3. Indeed, $\neg(\forall n.\, \delta n \downarrow \text{false})$ as there is a future where $\delta\, 42 \downarrow \text{true}$
4. However, $\exists n.\, \delta n \downarrow \text{true}$ is contradictory when $\delta n \downarrow \text{false}$ evolves for all n

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If we can control which functions have access to choice sequences,
this allows us to separate versions of Markov's principle!

Outline

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- General framework for type theories modelled through an abstract modality $\Box$ and parameterised by a type of time-progressing choice operators $\mathcal{C}$

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Constructing Separating Models using $\mathsf{TT}^{\Box}_{\mathcal{C}}$

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Every instance of $\mathsf{TT}_{\mathcal{C}}^{\square}$ interprets $\mathsf{CIC}^* = \mathsf{MLTT} + \mathbb{U} + ||\cdot||$, that is:

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Separating $MP_{\mathbb{B}}$ from MP_{PR}

Instance TT_1: choice sequences of type $\mathbb{N} \to \mathbb{B}$, first show $TT_1 \vdash \neg MP_{\mathbb{B}}$

world w

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Separating $MP_{\mathbb{B}}$ from MP_{PR} (ctd.)

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$$\frac{w \Vdash \Gamma, n : \mathbb{N} \cap \mathsf{pure} \vdash ||P\ n||}{w \Vdash \Gamma \vdash \forall n : \mathbb{N}.\, ||P\ n||}$$

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Thus for any countable sequence of pure functions $e : (\mathbb{N} \to (\mathbb{N} \to \mathbb{B})) \cap \mathsf{pure}$:

$$\forall w.\ w \Vdash \forall i : \mathbb{N}.\ \neg\neg||\Sigma n : \mathbb{N}.\ e_i\ n = \mathsf{true}|| \to ||\Sigma n : \mathbb{N}.\ e_i\ n = \mathsf{true}||$$

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In particular, we can set $e_i i$ to be the i-th primitive recursive function:

$$\forall w.\ w \Vdash MP_{PR} \quad \text{and therefore} \quad TT_1 \vdash MP_{PR}$$

Separating $\mathsf{MP}_{\mathbb{P}}$ from $\mathsf{MP}_{\mathbb{B}}$

Instance TT_2: choice sequences of type $\mathbb{N} \to \mathbb{U}$, follow same outline

Separating $MP_{\mathbb{P}}$ from $MP_{\mathbb{B}}$

Instance TT_2: choice sequences of type $\mathbb{N} \to \mathbb{U}$, follow same outline

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3. Exploit that $\mathbb{N} \to \mathbb{B}$ is now pure
4. Derive that $\mathrm{TT}_2 \vdash \mathrm{MP}_{\mathbb{B}}$ and $\mathrm{TT}_2 \vdash \neg\mathrm{MP}_{\mathbb{P}}$

Outline

Separating Limited Principles of Omniscience[1]

"Σ_1 propositions are logically decidable"

[1] da Rocha Paiva, Cohen, Forster, K., Rahli (2024)

Separating Limited Principles of Omniscience[1]

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\begin{aligned}
\mathsf{LPO}_{\mathbb{P}} &:= \forall p : \mathbb{N}\to\mathbb{P}.\, (\forall n.\, p\, n \vee \neg p\, n) \to && (\exists n.\, p\, n) \vee \neg(\exists n.\, p\, n)\\
\mathsf{LPO}_{\mathbb{B}} &:= \forall f : \mathbb{N}\to\mathbb{B}. && (\exists n.\, f\ n = \mathsf{true}) \vee \neg(\exists n.\, f\ n = \mathsf{true})\\
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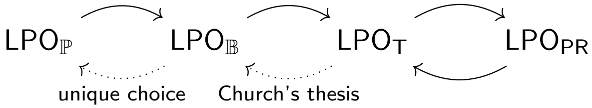

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Separating Arithmetical Fragments of LEM[2]

Represent the C-arithmetical hierarchy on predicates $p : \mathbb{N}^k \to \mathbb{P}$ inductively:

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Introduces hierarchy generalising MP_C and LPO_C:

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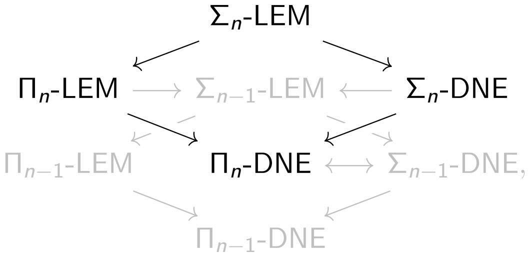

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Post's Hierarchy Theorem[3]

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Definition (Forster, K., Mück (2023))

An oracle computation is a functional $F : (Q \to A \to \mathbb{P}) \to I \to O \to \mathbb{P}$ captured by a computation tree $\tau : I \to A^* \rightharpoonup Q + O$ and its induced interrogation relation $\tau\, i ; R \vdash qs ; as$ as follows:

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Lemma

Assuming Σ_n-LEM, if P is Σ_{n+1} and Q is Σ_n, then $\mathcal{S}_Q(P)$.

[3] Forster, K., Mück (2024)

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[4] Zeng, Forster, K., Nemoto (2024)

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Definition (Shoenfield (1959) and Gold (1965))

$P : X \to \mathbb{P}$ is limit-computable if there exists a function $f : X \to \mathbb{N} \to \mathbb{B}$ with

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$P : X \to \mathbb{P}$ is low if $P' \preceq_T H$ and simple if it is co-infinite, semi-decidable, and for W_e being the e-th enumerable set we have $W_e \cap P \neq \emptyset$ whenever W_e is infinite.

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Theorem

Assuming LPO, a low simple set exists.

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Some Take-Home Messages

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Thank you!

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