(Sober Thoughts on) The Blurred Drinker Paradox[1]

Dominik Kirst, Haoyi Zeng

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[1]Forthcoming at LICS'25.

Constructive Reverse Mathematics

Classical reverse mathematics studies classically detectable equivalences:

- Which theorems are equivalent to the axiom of choice or similar principles?
- Which theorems are equivalent to which comprehension principles?
- Many more, see Friedman (1976) and Simpson (2009)

Constructive reverse mathematics studies constructively detectable equivalences:

- Which theorems are equivalent to excluded middle (LEM) or weaker principles?
- Which theorems are equivalent to which specific formulation of the axiom of choice?
- Many more, see Ishihara (2006) and Diener (2018)

Characterises the computational content of analysed theorems

The Downwards Löwenheim-Skolem Theorem[2]

Definition (Elementary Submodels)

Given first-order models $\mathcal{M}$ and $\mathcal{N}$, we call $h : \mathcal{M} \to \mathcal{N}$ an elementary embedding if

$$\forall \rho : \mathbb{N} \to \mathcal{M}. \forall \varphi. \mathcal{M} \vDash_\rho \varphi \leftrightarrow \mathcal{N} \vDash_{h \circ \rho} \varphi.$$

If such an elementary embedding h exists, we call $\mathcal{M}$ an elementary submodel of $\mathcal{N}$.

Theorem (DLS)

Every model has a countable elementary submodel.

What is the constructive status of the DLS theorem?

[2] Löwenheim (1915); Skolem (1920)

Classical Reverse Mathematics of DLS[3]

$$\begin{aligned}\mathrm{DC}_A &:= \forall R : A \to A \to \mathbb{P}.\, \mathrm{tot}(R) \to \exists f : \mathbb{N} \to A. \forall n.\, R\,(f\ n)\,(f\,(n+1))\\ \mathrm{CC}_A &:= \forall R : \mathbb{N} \to A \to \mathbb{P}.\, \mathrm{tot}(R) \to \exists f : \mathbb{N} \to A. \forall n.\, R\, n\,(f\ n)\end{aligned}$$

Theorem

The DLS *theorem is equivalent to* DC.

Sketch.

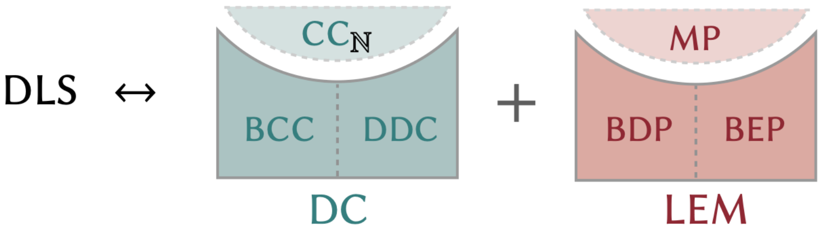
- To prove DLS from DC, arrange the iterative construction such that a single application of DC yields a path through all possible extensions that induces the resulting submodel.
- Starting with a total relation $R : A \to A \to \mathbb{P}$, consider (A, R) a model. Applying DLS, obtain an elementary submodel $(\mathbb{N}, R')$ so in particular R' is still total. Apply $\mathrm{CC}_{\mathbb{N}}$ to obtain a choice function for R' that is reflected back to A as a path through R. □

[3] Boolos et al. (2002); Espíndola (2012); Karagila (2014)

Constructive Reverse Mathematics of DLS?

Over a base theory like the Calculus of Inductive Constructions of the Coq Proof Assistant:

1. Does the DLS theorem still follow from DC alone or is there some contribution of LEM?
2. Does the DLS theorem still imply DC or is there some contribution of CC?



Classical Argument

DLS using Henkin Environments

Definition (Henkin Environment)

Given a model $\mathcal{M}$, we call $\rho : \mathbb{N} \to \mathcal{M}$ a Henkin environment if for all φ:

$$\exists n. (\mathcal{M} \vDash_\rho \varphi[\rho\, n] \;\rightarrow\; \mathcal{M} \vDash_\rho \dot{\forall}\varphi)$$
$$\exists n. (\mathcal{M} \vDash_\rho \dot{\exists}\varphi \;\rightarrow\; \mathcal{M} \vDash_\rho \varphi[\rho\, n])$$

Lemma

Every model with a Henkin environment has a countable elementary submodel.

Proof.

Given $\mathcal{M}$ and a Henkin environment ρ, we construct a countable elementary submodel $\mathcal{N}$ with domain $\mathbb{T}$ by $f^{\mathcal{N}}\,\vec{t} := f\,\vec{t}$ and $P^{\mathcal{N}}\,\vec{t} := P^{\mathcal{M}}\,(\hat{\rho}\,\vec{t})$. Then for the case $\dot{\forall}\varphi$ and Henkin witness n:

$$\mathcal{N} \vDash \dot{\forall}\varphi \;\Rightarrow\; \forall t.\, \mathcal{N} \vDash \varphi[t] \;\overset{\text{IH}}{\Rightarrow}\; \forall t.\, \mathcal{M} \vDash_\rho \varphi[\hat{\rho}\, t] \;\Rightarrow\; \mathcal{M} \vDash_\rho \varphi[\rho\, n] \;\Rightarrow\; \mathcal{M} \vDash_\rho \dot{\forall}\varphi$$

The Drinker Paradox

In every bar, one can identify a person such that,
if they drink, then the whole bar drinks

$$\mathsf{DP}_A \;:= \forall P : A \to \mathbb{P}.\, \exists x.\, (P\,x \to \forall y.\, P\,y)$$
$$\mathsf{EP}_A \;:= \forall P : A \to \mathbb{P}.\, \exists x.\, ((\exists y.\, P\,y) \to P\,x)$$

Fact (contrasting Warren and Diener (2018))

DP *and* EP *are equivalent to* LEM.

Proof.

To derive LEM from DP, given $p : \mathbb{P}$ use DP for $A := \{b : \mathbb{B} \mid b = \mathsf{false} \lor (p \lor \neg p)\}$ and $P : A \to \mathbb{P}$ defined by $P\,(\mathsf{true}, \_) := \neg p$ and $P\,(\mathsf{false}, \_) := \top$. $\square$

DLS assuming DC and LEM

Theorem

Assuming DC *and* LEM, *the* DLS *theorem holds.*

Proof.

Construct a Henkin environment in three steps:

1. Given some environment ρ, we know by DP and EP that, relative to ρ, Henkin witnesses for all formulas exist.
2. Applying CC we can simultaneously choose from these witnesses at once and therefore extend to some environment ρ'.
3. This describes a total relation on environments, through which DC yields a path that can be merged into a single environment, and that then must be Henkin. $\square$

Reverse Analysis

Theorem

Assuming $\mathsf{CC}_{\mathbb{N}}$, *the* DLS *theorem implies* DC.

Proof.

Following the outline from the beginning, using the assumption of $CC_{\mathbb{N}}$ to obtain a choice function in the countable elementary submodel. □

So over $\mathsf{CC}_{\mathbb{N}}$ and LEM, the DLS theorem is equivalent to DC.

Refining the Use of LEM

Recap: DLS using Henkin Environments

Definition (Henkin Environment)

Given a model $\mathcal{M}$, we call $\rho : \mathbb{N} \to \mathcal{M}$ a Henkin environment if for all φ:

$$\exists n.\,(\mathcal{M} \vDash_\rho \varphi[\rho\, n] \;\to\; \mathcal{M} \vDash_\rho \dot{\forall}\varphi)$$
$$\exists n.\,(\mathcal{M} \vDash_\rho \dot{\exists}\varphi \;\to\; \mathcal{M} \vDash_\rho \varphi[\rho\, n])$$

Lemma

Every model with a Henkin environment has a countable elementary submodel.

Proof.

Given $\mathcal{M}$ and a Henkin environment ρ, we construct a countable elementary submodel $\mathcal{N}$ with domain $\mathbb{T}$ by $f^{\mathcal{N}}\,\vec{t} := f\,\vec{t}$ and $P^{\mathcal{N}}\,\vec{t} := P^{\mathcal{M}}\,(\hat{\rho}\,\vec{t})$. Then for the case $\dot{\forall}\varphi$ and Henkin witness n:

$$\mathcal{N} \vDash \dot{\forall}\varphi \;\Rightarrow\; \forall t.\,\mathcal{N} \vDash \varphi[t] \;\overset{\text{IH}}{\Rightarrow}\; \forall t.\,\mathcal{M} \vDash_\rho \varphi[\hat{\rho}\,t] \;\Rightarrow\; \mathcal{M} \vDash_\rho \varphi[\rho\, n] \;\Rightarrow\; \mathcal{M} \vDash_\rho \dot{\forall}\varphi$$

DLS using Blurred Henkin Environments

Definition (Henkin Environment)

Given a model $\mathcal{M}$, we call $\rho : \mathbb{N} \to \mathcal{M}$ a blurred Henkin environment if or all φ:

$$(\forall n.\, \mathcal{M} \vDash_\rho \varphi[\rho\, n]) \;\to\; \mathcal{M} \vDash_\rho \dot{\forall}\varphi$$
$$\mathcal{M} \vDash_\rho \dot{\exists}\varphi \;\to\; (\exists n.\, \mathcal{M} \vDash_\rho \varphi[\rho\, n])$$

Lemma

Every model with a blurred Henkin environment has a countable elementary submodel.

Proof.

Given $\mathcal{M}$ and a Henkin environment ρ, we construct a countable elementary submodel $\mathcal{N}$ with domain $\mathbb{T}$ by $f^{\mathcal{N}}\,\vec{t} := f\,\vec{t}$ and $P^{\mathcal{N}}\,\vec{t} := P^{\mathcal{M}}\,(\hat{\rho}\,\vec{t})$. Then for the case $\dot{\forall}\varphi$ and Henkin witness n:

$$\mathcal{N} \vDash \dot{\forall}\varphi \;\Rightarrow\; \forall t.\, \mathcal{N} \vDash \varphi[t] \;\overset{\mathsf{IH}}{\Rightarrow}\; \forall t.\, \mathcal{M} \vDash_\rho \varphi[\hat{\rho}\, t] \;\Rightarrow\; \forall n.\, \mathcal{M} \vDash_\rho \varphi[\rho\, n] \;\Rightarrow\; \mathcal{M} \vDash_\rho \dot{\forall}\varphi$$

The Blurred Drinker Paradox (BDP)

In every bar, there is an at most countable group such that,
if all of them drink, the the whole bar drinks

$$\begin{aligned} \mathrm{BDP}_A &:= \forall P : A \to \mathbb{P}. \exists f : \mathbb{N} \to A. (\forall y. P(f\,y)) \to \forall x. P\,x \\ \mathrm{BEP}_A &:= \forall P : A \to \mathbb{P}. \exists f : \mathbb{N} \to A. (\exists x. P\,x) \to \exists y. P(f\,y) \end{aligned}$$

Fact

LEM *decomposes into* BDP + $\mathrm{DP}_{\mathbb{N}}$ *and even* BDP + MP, *similarly for* BEP.

Proof.

The first decomposition is trivial. The latter follows since BDP implies Kripke's schema (KS) which is known to imply LEM in connection to MP. □

Classification of BDP

$$
\begin{tikzcd}
 & & \mathrm{LEM}/\mathrm{DP}/\mathrm{EP}/\mathrm{IP} \arrow[dll] \arrow[d] \arrow[drr] & & \\
\mathrm{BDP} \arrow[dd] & & \mathrm{LPO} \arrow[dd] & & \mathrm{BEP} \arrow[dd] \\
 & & & & \\
\mathrm{KS}' \arrow[r, dash, dashed] & {} \arrow[r, dash, dashed] \arrow[uuur, dashed, bend left] & \mathrm{MP} \arrow[r, dash, dashed] & {} \arrow[r, dash, dashed] \arrow[uuul, dashed, bend right] & \mathrm{KS}
\end{tikzcd}
$$

DLS assuming DC and BDP

Theorem

Assuming DC *and* BDP/BEP, *the* DLS *theorem holds.*

Proof.

Construct a blurred Henkin environment in three steps:

1. Given some environment ρ, we know by BDP/BEP that, relative to ρ, blurred Henkin witnesses for all formulas exist.
2. Applying CC we can simultaneously choose from these witnesses at once and therefore extend to some environment ρ'.
3. This describes a total relation on environments through which DC yields a path, that can be merged into a single environment, and that then must be blurred Henkin. □

Reverse Analysis

Theorem

The DLS *theorem implies* BDP *and* BEP*.*

Proof.

Using the same pattern as in the previous analysis, basically DLS reduces BDP to the trivially provable $\mathrm{BDP}_{\mathbb{N}}$, respectively BEP to the trivially provable $\mathrm{BEP}_{\mathbb{N}}$. $\square$

So over $\mathrm{CC}_{\mathbb{N}}$, the DLS theorem decomposes into DC+BDP+BEP.

Refining the Use of DC

Blurred Choice Axioms[4]

$$\mathrm{BCC}_A \;:=\; \forall R : \mathbb{N} \to A \to \mathbb{P}.\, \mathrm{tot}(R) \to \exists f : \mathbb{N} \to A. \forall n. \exists m.\, R\, n\, (f\ m)$$
$$\mathrm{DDC}_A \;:=\; \forall R : A \to A \to \mathbb{P}.\, \mathrm{dir}(R) \to \exists f : \mathbb{N} \to A.\, \mathrm{dir}(R \circ f)$$

Lemma

CC *decomposes into* $\mathrm{BCC} + \mathrm{CC}_{\mathbb{N}}$ *and* DC *decomposes into* $\mathrm{DDC} + \mathrm{CC}$.

$$\mathrm{BDC}^2_A \;:=\; \forall R : A^2 \to A \to \mathbb{P}.\, \mathrm{tot}(R) \to \exists f : \mathbb{N} \to A.\, \mathrm{tot}(R \circ f)$$

Lemma

BDC^2 *decomposed into* $\mathrm{BCC} + \mathrm{DDC}$, *so* DC *decomposes into* $\mathrm{BDC}^2 + \mathrm{CC}_{\mathbb{N}}$.

[4] An instance of BCC was independently observed by CASTRO (2024).

Classification of Blurred Choice Axioms

$$
\begin{tikzcd}
\mathrm{DC} \arrow[rr] \arrow[d] & & \mathrm{CC} \arrow[d] \\
\mathrm{BDC}^2 \arrow[r] \arrow[d] & \mathrm{BDC} \arrow[r] & \mathrm{BCC} \\
\mathrm{DDC} & &
\end{tikzcd}
$$

DLS assuming BDC and BDP

Theorem

Assuming BDC^2 *and* $\mathsf{BDP}/\mathsf{BEP}$, *the* DLS *theorem holds.*

Proof.

Construct a blurred Henkin environment in three steps:

1. Given some environment ρ, we know by BDP/BEP that, relative to ρ, blurred Henkin witnesses for all formulas exist.
2. Applying BCC we can simultaneously choose from these witnesses at once and therefore extend to some environment ρ'.
3. This describes a directed relation on environments, through which DDC yields a path that can be merged into a single environment, and that then must be blurred Henkin. □

Reverse Analysis

Theorem

The DLS *theorem implies* BDC^2 *and therefore also* BCC *and* DDC*.*

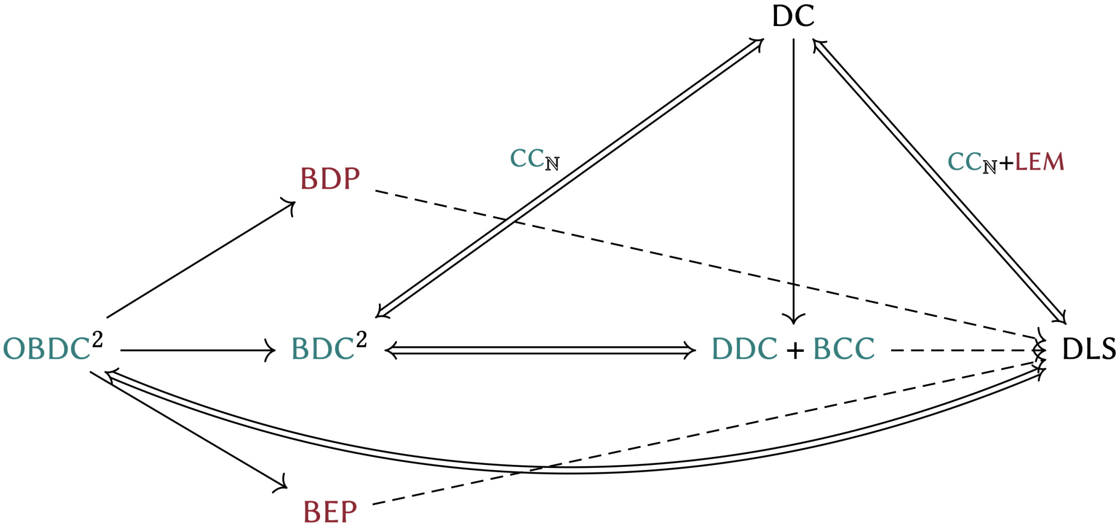
Proof.

Using the same pattern as in the previous analyses. □

So the DLS theorem decomposes into $\mathrm{BDC}^2 + \mathrm{BDP} + \mathrm{BEP}$.

Conclusion

Overview



Does any of this actually make sense?

Is it clear that BDP does not imply DP?

- At least we don't have BDP_A implying DP_A for the choice $A := \mathbb{N}$
- Is there a model satisfying all of BDP but not DP?

Is it clear that BCC does not imply CC?

- Again, at least we don't have BCC_A implying CC_A for the choice $A := \mathbb{N}$
- Seems like non-deterministic realisability satisfies BCC but refutes $\mathrm{CC}_{\mathbb{N}}$

How about DDC $\not\to$ BDC^2, DDC $\not\to$ BCC, BDP $\not\to$ BCC, BCC $\not\to$ BDP, etc.?

BDP as a Natural Hierarchy of Principles Below LEM

$$\mathrm{BDP}_A^B := \forall P : A \to \mathbb{P}. \exists f : B \to A. (\forall y. P(f\,y)) \to \forall x. P\,x$$
$$\mathrm{BEP}_A^B := \forall P : A \to \mathbb{P}. \exists f : B \to A. (\exists x. P\,x) \to \exists y. P(f\,y)$$

Fact

1. *Both* BDP_A^A *and* BEP_A^A *are provable.*
2. *If* BDP_A^B *and* BDP_B^C, *then* BDP_A^C.
3. *If* BEP_A^B *and* BEP_B^C, *then* BEP_A^C.
4. DP_A *implies* BDP_A^B *and is equivalent to* $\mathrm{BDP}_A^{\mathbb{1}}$.
5. EP_A *implies* BEP_A^B *and is equivalent to* $\mathrm{BEP}_A^{\mathbb{1}}$.
6. $\mathrm{BDP}_A^{\mathbb{B}}$ *is strictly weaker than* DP_A *but* $\mathrm{BDP}^{\mathbb{B}}$ *still implies* LEM.

Remaining Questions?

- Are the blurred principles weaker than the original?
  - We expect BDP $\not\to$ LEM, BCC $\not\to$ CC, and DDC $\not\to$ BCC
- What happens with uncountable cardinalities?
  - Weaker forms of blurred drinker paradoxes, stronger forms of blurred choice principles
- What is the constructive status of the upwards Löwenheim-Skolem theorem?
  - Usual proof strategy uses compactness which is as non-constructive as completeness

Thank you!

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