Constructive and Mechanised Meta-Theory of IEL and Similar Modal Logics

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Outline

Talk designed for 1h, if you have questions please interrupt any time!

Outline with Pointers

1. Intuitionistic Epistemic Logic (Artemov and Protopopescu, 2016)
2. Constructive Completeness of IEL (Hagemeier and Kirst, 2022b)
3. Applicability to other Modal Logics (Hagemeier and Kirst, 2022a)
4. Constructive Reverse Mathematics of Completeness (Kirst, 2022)
5. Conclusion

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Intuitionistic Epistemic Logic (IEL)

Classical epistemic logic (Hintikka, 1962)

- Extend classical logic with modality K
- Add axioms for K capturing understanding of belief/knowledge
- Reflection principle $\mathsf{K}\,A \to A$: "Known propositions are true"

Intuitionistic epistemic logic (Artemov and Protopopescu, 2016)

- Understand truth as intuitionistic provability (BHK-interpretation)
- Co-reflection principle $A \to \mathsf{K}\,A$: "From proofs we gain knowledge by verification"
- Intuitionistic reflection $\mathsf{K}\,A \to \neg\neg A$: "Known propositions are potentially true"

$\mathsf{IEL}^- := \mathsf{IPC} + \mathsf{K} + \text{co-reflection}$ $\qquad$ $\mathsf{IEL} := \mathsf{IEL}^- + \text{intuitionistic reflection}$

Meta-Theory of IEL

Artemov and Protopopescu (2016)

- Soundness and completeness with respect to suitable Kripke semantics
- Derived results: disjunction property, admissibility of reflection, etc.

Su and Sano (2019)

Finite model property and semantic cut-elimination

Krupski (2020)

Syntactic cut-elimination and decidability

Classical Meta-Theory of IEL

Fact

If $\mathcal{T} \Vdash A$ implies $\mathcal{T} \vdash A$ for arbitrary $\mathcal{T}$, then double-negation elimination holds.

Proof.

Given some proposition P and assuming $\neg\neg P$, consider $\mathcal{T} := \{A \in \mathcal{F} \mid P\}$. It is enough to show $\mathcal{T} \vdash \bot$, since then $\mathcal{T}$ must be non-empty and thus P holds. Apply completeness and show $\mathcal{T} \Vdash \bot$, so assume a model $\mathcal{M} \Vdash \mathcal{T}$ and derive a contradiction. Since we have $\neg\neg P$, on deriving a contradiction we may assume P. But then $\mathcal{M} \Vdash \bot$, contradiction. □

Fact

If $\mathcal{T} \Vdash A$ implies $\mathcal{T} \vdash A$ for enumerable $\mathcal{T}$, then Markov's principle (MP) holds.

Proof.

Let $f : \mathbb{N} \to \mathbb{B}$ with $\neg\neg(\exists n.\, f\ n = \text{true})$ be given. Using the enumerable set $\mathcal{T} := \{A \in \mathcal{F} \mid \exists n.\, f\ n = \text{true}\}$ derive $\exists n.\, f\ n = \text{true}$ with an argument as above. □

A General Observation

In any "usual" logic with $\bot$, completeness is connected to double-negation elimination:

Observation.

Suppose an arbitrary logic with a notion of models interpreting $\bot$ with meta-level falsity. Assuming $\mathcal{T} \vDash A$ implies $\mathcal{T} \vdash A$ for $\mathcal{T}$ of complexity $\mathcal{S}$, one can derive double-negation elimination for propositions of complexity $\mathcal{S}$. □

Justification.

Same as before. Let P have complexity $\mathcal{S}$ and assume $\neg\neg P$. Exploit $\mathcal{S}$-completeness for the theory $\mathcal{T} := \{A \in \mathcal{F} \mid P\}$ with $\mathcal{T} \vDash \bot$ to derive $\mathcal{T} \vdash \bot$ and thus P as desired. □

To sidestep this effect, we later analyse quasi-completeness: $\mathcal{T} \Vdash A$ implies $\neg\neg(\mathcal{T} \vdash A)$

Does quasi-completeness hold constructively? Is enumerable completeness equivalent to MP?

Constructive Meta-Theory of IEL

Can IEL be meaningfully described in a constructive system?

Work in the constructive type theory CIC (Coquand and Huet, 1988; Paulin-Mohring, 1993):

- Expressive system implementing higher-order intuitionistic logic
- Clean analysis without obscuring choice principles (Richman, 2001; Forster, 2022)
- Obtain (variants of) main results without appeal to additional axioms

Fact (CIC models IEL)

The truncation operation $||X||$ squashing a computational type X of CIC into the propositional universe $\mathbb{P}$ satisfies co-reflection $X \to ||X||$ and intuitionistic reflection $||X|| \to \neg\neg X$.

Mechanised Meta-Theory of IEL[1]

Can IEL be feasibly mechanised in a proof assistant?

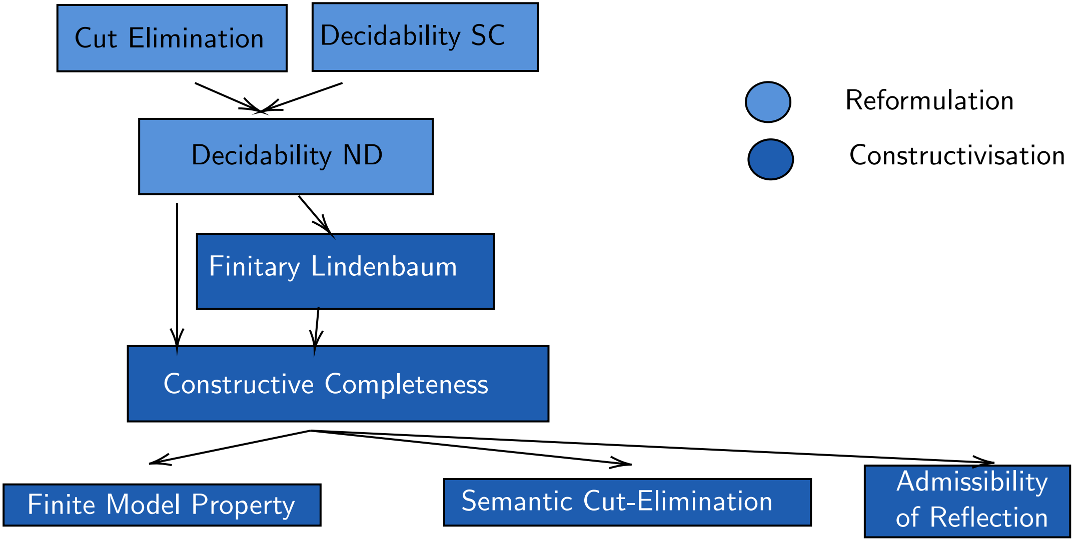
Work with the Coq proof assistant:

- Implements CIC, used as tool to verify our proofs and track assumptions
- Executable algorithms via constructive completeness, cut-elimination, and decidability
- Synthetic computability as a shortcut (Richman, 1983; Bauer, 2006; Forster et al., 2019)
- Development systematically hyperlinked with the papers

[1] `https://www.ps.uni-saarland.de/extras/iel-ext/`

Outline

Results Overview



Deduction Systems for IEL

Model deduction systems as inductive predicates of type $\mathcal{L}(\mathcal{F}) \to \mathcal{F} \to \mathbb{P}$.

Natural Deduction (ND)
Extends natural deduction for IPC by 3 rules (co-reflection, distribution and int. reflection)

$$\frac{\Gamma \vdash A}{\Gamma \vdash \mathsf{K}\,A}\ (KR) \qquad \frac{\Gamma \vdash \mathsf{K}\,(A \supset B)}{\Gamma \vdash \mathsf{K}\,A \supset \mathsf{K}\,B}\ (KD)$$

$$\frac{\Gamma \vdash \mathsf{K}\,A}{\Gamma \vdash \neg\neg A}\ (KF)$$

Sequent Calculus (SC)
Extend G3I by 2 rules (Krupski, 2020); we use GKI as base (better for mechanisation)

$$\frac{\Gamma \cup \{A \mid \mathsf{K}\,A \in \Gamma\} \Rightarrow B}{\Gamma \Rightarrow \mathsf{K}\,B}\ (\mathrm{KI})$$

$$\frac{\Gamma \Rightarrow \mathsf{K}\,\bot}{\Gamma \Rightarrow A}\ (\mathrm{KF})$$

In contrast to ND, SC is analytic, i.e. (almost) has the subformula property.

Cut-Elimination

Theorem (Cut-Elimination)

If $\Gamma \Rightarrow A$ *and* $\Gamma, A \Rightarrow B$ *then* $\Gamma \Rightarrow B$*.*

Proof.

Typical double induction on rank and size of a cut (cf. Troelstra/Schwichtenberg(2000)). □

Corollary (Agreement)

$\Gamma \vdash A$ *if and only if* $\Gamma \Rightarrow A$*.*

Proof.

Both directions are proven by induction on the given derivations; only direction from ND to SC needs Cut-Elimination. □

Decidability

Lemma

One can construct a function $f : \mathcal{F} \to \mathbb{B}$ such that $f\ A = \mathrm{true}$ if and only if $\Rightarrow A$.

- Synthetic notion of decidability (no Turing-machines; f computable by construction)
- Utilise subformula property of sequent calculus for IEL
- Compute derivable sequents as a fixed point of stepwise derivation

Theorem (Decidability)

SC and ND are decidable.

Proof.

By the previous lemma and the agreement of ND and SC. □

Lindenbaum Construction

Let $\mathcal{U}$ be finite and subformula-closed.

Definition (Primeness)

A set of formulas Γ is $\mathcal{U}$-prime $A \vee B \in \Gamma$ implies that $A \in \Gamma$ or $B \in \Gamma$ for all $A, B \in \mathcal{U}$.

Lemma

For any context $\Gamma \subseteq \mathcal{U}$ and formula $A_\perp$, we can compute Δ extending Γ which is $\mathcal{U}$-prime, closed under derivability in $\mathcal{U}$, and preserves non-derivability of $A_\perp$.

Proof.

Iterate through the formulas A_i of $\mathcal{U}$ to obtain contexts Γ_i. In step i, add A_i, if non-derivability of $A_\perp$ is preserved by the addition (using decidability):

$$\Gamma_{i+1} := \begin{cases} \Gamma_i, A_i & \text{if } \Gamma_i, A_i \nvdash A_\perp \\ \Gamma_i & \text{otherwise} \end{cases}$$

□

Decidable Universal Model

Given $\mathcal{U}$, build a canonical Kripke model $\mathcal{M}_{\mathcal{U}} = (\mathcal{W}_{\mathcal{U}}, \mathcal{V}_{\mathcal{U}}, \leq, \leq_{\mathsf{K}})$:

- $\mathcal{W}_{\mathcal{U}}$ contains $\mathcal{U}$-prime, consistent $\mathcal{U}$-theories as worlds
- $\mathcal{V}_{\mathcal{U}}(\Gamma, i) := p_i \in \Gamma$
- $\Gamma \leq \Delta := \Gamma \subseteq \Delta$
- $\Gamma \leq_{\mathsf{K}} \Delta := \Gamma \cup \{A \mid \mathsf{K}\, A \in \Gamma\} \subseteq \Delta$ (same as in Su and Sano (2019b))

Lemma (Truth Lemma)

For $A \in \mathcal{U}$ and $\Gamma \in \mathcal{W}_{\mathcal{U}}$, we have $A \in \Gamma \iff \Gamma \Vdash A$.

Proof.

Induction on A. Using decidability of membership and the Lindenbaum Lemma. □

Theorem (Finitary Completeness)

If $\Vdash A$ then $\vdash A$, or equivalently, if $\Gamma \Vdash A$ then $\Gamma \vdash A$ for finite Γ.

Proof.

Finite Model Property

Definition (FMP)

IEL has FMP, if $\vdash A$ whenever $\mathcal{M} \Vdash A$ for all (essentially) finite $\mathcal{M}$.

Theorem

IEL has the finite model property.

Proof.

Given the bound against $\mathcal{U}$, the canonical model is (essentially) finite. □

Semantic Cut-Elimination[2]

Lemma (Completeness SC)

If $\Gamma \Vdash A$ *then* $\Gamma \Rightarrow A$.

Proof.

Canonical model construction with respect to SC using saturated theories. □

Theorem (SCE)

If $\Gamma \vdash A$ *then* $\Gamma \Rightarrow A$.

Proof.

By composition of Soundness and Completeness. □

[2] Following Su and Sano (2019a)

Outline

Other logics

- Obvious question: Does the method apply to other logics (eg. classical, first-order)?
- Impossible for FOL
- Partial answer: Some classical propositional modal logics work

Modal Logic K

- Simplest propositional modal logic
- Treat $\Diamond$ as derived modality (in code: same datatype but impossible to reduce to two logical constants)
- We try to use similar strategy (i.e. explicitly constructed canonical model)

Natural deduction for K

- Somewhat unusual choice. (Nicer for mechanisation, more experience)
- Enrich natural deduction for classical logic with following rules:

$$\frac{\neg A, \Gamma \vdash^{K} \bot}{\Gamma \vdash^{K} A}\ (\textsc{E}) \qquad \frac{\vdash^{K} A}{\Gamma \vdash^{K} \Box(A)}\ (\textsc{Nec}) \qquad \frac{\Gamma \vdash^{K} \Box(A \supset B)}{\Gamma \vdash^{K} \Box A \supset \Box B}\ (\textsc{Dist})$$

- Provably equivalent to known axiomatisations of K

Decidability for K

- Introduce cut-free sequent calculus (based on G3c) inspired by Hakli and Negri (2012), then same strategy as for IEL.
- Classical logic, thus sets of formulas on both sides ($\Rightarrow^K : \mathcal{L}(\mathcal{F}) \to \mathcal{L}(\mathcal{F}) \to \mathbb{P}$)

$$\frac{\Box A \in \Omega \qquad \Gamma_\Box \Rightarrow^K A}{\Gamma \Rightarrow^K \Omega} \quad (\mathrm{K})$$

Theorem

NDK is deciable.

Proof.

Combine agreement between SCK and NDK (cut-elimination) and decider for SCK using fixed-point iteration. □

Lindenbaum for K

- Classical Lindenbaum construction always adds either a formula or its negation!
- A context Γ is $\mathcal{U}$-maximal if for any $A \in \mathcal{U}$ we have $A \in \Gamma$ or $\neg A \in \Gamma$.
- Thus, need a bigger subformula universe.

$$\mathcal{U}^* := \mathcal{U} \cup \neg(\mathcal{U})$$

- Need to be careful, which statements are formulated w.r.t. $\mathcal{U}$ or $\mathcal{U}^*$.

Lemma (Lindenbaum Lemma)

For any context $\Gamma \subseteq \mathcal{U}^*$ and formula $A_\perp$, we can compute Δ extending Γ which is prime, consistent theory that is $\mathcal{U}$-maximal and preserves non-derivability of $A_\perp$.

Canonical model for K

Definition (Canonical Model)

We define $\mathcal{M}_C = (\mathcal{W}_C, \mathcal{V}_C, \leq)$ by

- $\mathcal{W}_C := \{\Gamma \subseteq \mathcal{U}^* \mid \Gamma$ is a $\mathcal{U}$-maximal, prime, consistent list of formulas$\}$
- $\mathcal{V}_C(\Gamma, i) := p_i \in \Gamma$
- $\Gamma \leq \Delta := \Gamma_\Box \subseteq \Delta$

Lemma (Truth Lemma K)

For any $\Gamma \in \mathcal{W}_C$ and $A \in \mathcal{U}^*$ we have

$$A \in \Gamma \iff \mathcal{M}_C, \Gamma \Vdash A.$$

Theorem (Finitary Completeness (K))

If $\Vdash A$ then $\vdash^K A$.

The modal logic cube

- What about other modal logics?
- Provisio: Assume constructive decider (Wu and Goré, 2019)
- Can use same strategy for KD, KT
- Had no success for stronger other modal logics (e.g. containing 4 axiom)

$$\mathsf{D} := \Box A \to \Diamond A \qquad \mathsf{T} := \Box A \to A \qquad 4 := \Box A \to \Box\Box A$$

KD and KT

- D corresponds to seriality, T to reflexivity
- Only need to establish that canonical modal has frame condition
- We too, mirror the proof and show that the canonical model is serial

Theorem (Canonical model for D is serial)

The canonical model for D is serial.

Proof.

Let Γ be a world in the canonical model for D. First, notice that $\Gamma \nvdash \Box\bot$. Thus we can Lindenbaum-extend $\Gamma_\Box$ to a sucessor.
(Assume $\Gamma \vdash \Box\bot$. Thus derive $\Gamma \vdash \Diamond\bot$ by D axiom. However $\neg\Diamond\bot$ is a theorem of D. Contradiction.) □

Theorem (Canonical model for T is reflexive)

The canonical model for T is reflexive.

Outline

Analysing Completeness Theorems in Constructive Meta-Theory

Confusing situation in the literature on first-order logic:

- Completeness equivalent to Boolean Prime Ideal Theorem (Henkin, 1954)
- Completeness requires Markov's Principle (Kreisel, 1962)
- Completeness equivalent to Weak Kőnig's Lemma (Simpson, 2009)
- Completeness holds fully constructively (Krivine, 1996)

Systematic investigation missing:

- Started consolidation by Herbelin and Ilik (2016), Forster et al. (2021), and Kirst (2022)
- Comprehensive overview of current landscape by Herbelin (2022)

Classical Completeness Proof

Typical outline for IEL (same for IPC and others):

- Lindenbaum Extension: if $\mathcal{T} \nvdash A$ then there is prime $\mathcal{T}'$ with $\mathcal{T}' \nvdash A$
- Universal Model: consistent prime theories related by inclusion
- Truth Lemma: $A \in \mathcal{T} \iff \mathcal{T} \Vdash A$
- Model Existence: if $\mathcal{T} \nvdash A$ then there is $\mathcal{M}$ with $\mathcal{M} \Vdash \mathcal{T}$ and $\mathcal{M} \nVdash A$
- Quasi-Completeness: if $\mathcal{T} \Vdash A$ then $\neg\neg(\mathcal{T} \vdash A)$
- Completeness: if $\mathcal{T} \Vdash A$ then $\mathcal{T} \vdash A$

Constructive Completeness Proof???

For $\mathcal{T}$ quasi-prime ($A \vee B \in \mathcal{T} \to \neg\neg(A \in \mathcal{T} \vee A \in \mathcal{T})$):

- Lindenbaum Extension: if $\mathcal{T} \nvdash A$ then there is quasi-prime $\mathcal{T}'$ with $\mathcal{T}' \nvdash A$
- Universal Model: consistent quasi-prime theories related by inclusion
- Truth Lemma: fails immediately
- Model Existence: fails
- Quasi-Completeness: fails
- Completeness: anyway no constructive consequence of quasi-completeness

Constructive Completeness Proof?

For $\mathcal{T}$ quasi-prime ($A \lor B \in \mathcal{T} \to \neg\neg(A \in \mathcal{T} \lor A \in \mathcal{T})$) and stable ($\neg\neg(A \in \mathcal{T}) \to A \in \mathcal{T}$):

- Lindenbaum Extension: if $\mathcal{T} \nvdash A$ then there is stable quasi-prime $\mathcal{T}'$ with $\mathcal{T}' \nvdash A$
- Universal Model: consistent stable quasi-prime theories related by inclusion
- Truth Lemma: fails for disjunction
- Model Existence: fails
- Quasi-Completeness: fails
- Completeness: anyway no constructive consequence of quasi-completeness

The Issue with Disjunction

Truth Lemma case for disjunctions $A \vee B$:

$$\begin{aligned} A \vee B \in \mathcal{T} &\overset{?}{\iff} \mathcal{T} \Vdash A \vee B \\ &\overset{def}{\iff} \mathcal{T} \Vdash A \vee \mathcal{T} \Vdash B \\ &\overset{IH}{\iff} A \in \mathcal{T} \vee B \in \mathcal{T} \end{aligned}$$

- So we really need prime theories for disjunctions
- Primeness from Lindenbaum Extension is constructive no-go

Quasi-Completeness via WLEM

Weak law of excluded middle WLEM $:= \forall P : \mathbb{P}. \neg P \vee \neg\neg P$

Lemma

Assuming WLEM, every stable quasi-prime theory is prime.

Proof.

Assume $A \vee B \in \mathcal{T}$. Using WLEM, decide whether $\neg(A \in \mathcal{T})$ or $\neg\neg(A \in \mathcal{T})$. In the latter case, conclude $A \in \mathcal{T}$ directly by stability. In the former case, derive $B \in \mathcal{T}$ using stability, since assuming $\neg(B \in \mathcal{T})$ on top of $\neg(A \in \mathcal{T})$ contradicts quasi-primeness for $A \vee B \in \mathcal{T}$. □

Classical proof outline works again up to quasi-completeness!

What happens if we instead weaken the Truth Lemma?

Quasi-Completeness via DNS

Assuming double-negation shift DNS $:= \forall X. \forall p : X \to \mathbb{P}. (\forall x. \neg\neg p\,x) \to \neg\neg(\forall x. p\,x)$:

- Lindenbaum Extension: if $\mathcal{T} \nvdash A$ then there is stable quasi-prime $\mathcal{T}'$ with $\mathcal{T}' \nvdash A$
- Universal Model: consistent stable quasi-prime theories related by inclusion
- Quasi Truth Lemma: $A \in \mathcal{T} \iff \neg\neg(\mathcal{T} \Vdash A)$
- Quasi Model Existence: if $\mathcal{T} \nvdash A$ then there is $\mathcal{M}$ with $\neg\neg(\mathcal{M} \Vdash \mathcal{T})$ and $\mathcal{M} \nVdash A$
- Quasi-Completeness: if $\mathcal{T} \Vdash A$ then $\neg\neg(\mathcal{T} \vdash A)$ (also since DNS $\iff \neg\neg$LEM)
- Completeness: anyway no constructive consequence of Quasi-Completeness

Backwards Analysis

Two proofs of Quasi-Completeness from incomparable principles...

Fact

Model Existence implies WLEM.

Proof.

Given P, use model existence on $\mathcal{T} := \{x_0 \lor \neg x_0\} \cup \{x_0 \mid P\} \cup \{\neg x_0 \mid \neg P\}$. We have $\mathcal{T} \nvdash \bot$ so if $\mathcal{M} \Vdash \mathcal{T}$, then either $\mathcal{M} \Vdash x_0$ or $\mathcal{M} \Vdash \neg x_0$, so either $\neg\neg P$ or $\neg P$, respectively. □

Fact

Quasi-Completeness implies the following principle: $\forall p : \mathbb{N} \to \mathbb{P}. \neg\neg(\forall n. \neg p\, n \lor \neg\neg p\, n)$

Proof.

Using similar tricks for $\mathcal{T} := \{x_n \lor \neg x_n\} \cup \{x_n \mid p\, n\} \cup \{\neg x_n \mid \neg p\, n\}$, see backup slide. □

Obvious consequence both from WLEM and DNS, maybe enough for Quasi-Completeness?

Weak Double-Negation Shift (Preliminary Name)

$$\text{WDNS} \;:=\; \forall p : \mathbb{N} \to \mathbb{P}.\, \neg\neg(\forall n.\, \neg p\, n \vee \neg\neg p\, n)$$

Lemma

Assuming WDNS, every stable quasi-prime theory is not not prime.

Proof.

Assume $\mathcal{T}$ not prime and derive a contradiction. Given the negative goal, from WDNS we obtain $\forall A.\, \neg(A \in \mathcal{T}) \vee \neg\neg(A \in \mathcal{T})$. This yields exactly the instances of WLEM needed to derive that $\mathcal{T}$ is prime, contradiction. □

WDNS turns stable predicates $p : \mathbb{N} \to \mathbb{P}$ not not decidable, contributes to Fan Theorem

Already the Lemma turns out to be enough for Quasi-Completeness!

Quasi-Completeness via WDNS

Refined proof outline using WDNS:

- Lindenbaum Extension: if $\mathcal{T} \nvdash A$ then there is stable not not prime $\mathcal{T}'$ with $\mathcal{T}' \nvdash A$
- Universal Model: consistent stable prime theories related by inclusion
- Truth Lemma: $A \in \mathcal{T} \iff \mathcal{T} \Vdash A$
- Pseudo Model Existence: if $\mathcal{T} \nvdash A$ then there not not is $\mathcal{M}$ with $\mathcal{M} \Vdash \mathcal{T}$ and $\mathcal{M} \nVdash A$
- Quasi-Completeness: if $\mathcal{T} \Vdash A$ then $\neg\neg(\mathcal{T} \vdash A)$
- Completeness: anyway no constructive consequence of Quasi-Completeness

Consequences and Generalisation

Consequences:

- WLEM and Model Existence are equivalent
- WDNS, Pseudo Model Existence, and Quasi-Completeness are equivalent
- Completeness of IEL regarding enumerable $\mathcal{T}$ is equivalent to WDNS + MP

Generalisation:

- Classical and intuitionistic propositional logic
- Classical and intuitionistic modal logics
- Classical first-order logic, maybe intuitionistic first-order logic

Outline

Coq Mechanisation[3]

- Roughly 4k lines of code, structured in accordance with the papers
- Uses helpful features of Coq: e.g. can prove most results simultaneously for IEL and IEL$^-$ using a type class flag
- Method for mechanising syntactic results (i.e. decidability and cut-elimination) generalises to other logics, we instantiated to classical modal logic K, KD, and KT

Component	Spec	Proof
preliminaries	121	93
natural deduction + lindenbaum	183	418
models	43	23
completeness	75	325
semantic cut-elimination	49	214
cut-elimination + decidability IEL	193	399
classical completeness / infinite theories	90	261
cut-elimination + decidability K	116	362
completeness K	165	397
completeness argument T, D	290	625
$\sum$	1107	3181

Figure: Overview of the mechanisation components

[3] https://www.ps.uni-saarland.de/extras/iel-ext/

Conclusion

- Background: IEL is a convincing rendering of knowledge in intuitionistic framework
- Contribution: IEL has a well-behaved meta-theory in intuitionistic framework
- Method: Proof assistant helps ensuring correctness and exhibits algorithms
- Future Work: Systematic constructive reverse mathematics of completeness theorems

Thank You!

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SC

$$\frac{p_i \in \Gamma}{\Gamma \Rightarrow p_i} \qquad \frac{\bot \in \Gamma}{\Gamma \Rightarrow S} \qquad \frac{F, \Gamma \Rightarrow G}{\Gamma \Rightarrow F \supset G} \qquad \frac{F \supset G \in \Gamma \qquad \Gamma \Rightarrow F}{\Gamma \Rightarrow G}$$

$$\frac{F \wedge G \in \Gamma \qquad F, G, \Gamma \Rightarrow H}{\Gamma \Rightarrow H} \qquad \frac{\Gamma \Rightarrow F \qquad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \wedge G}$$

$$\frac{F \vee G \in \Gamma \qquad F, \Gamma \Rightarrow H \qquad G, \Gamma \Rightarrow H}{\Gamma \Rightarrow H} \qquad \frac{\Gamma \Rightarrow F_i}{\Gamma \Rightarrow F_1 \vee F_2} \qquad \frac{\Gamma \cup \Gamma_{\mathsf{K}} \Rightarrow F}{\Gamma \Rightarrow \mathsf{K}\, F}$$

ND

$$\frac{A \in \Gamma}{\Gamma \vdash A}\ \mathrm{A} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ \mathrm{E}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\ \mathrm{II} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash A \supset B}{\Gamma \vdash B}\ \mathrm{IE}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \mathrm{DIL} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \mathrm{DIR} \quad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C \quad \Gamma \vdash A \vee B}{\Gamma \vdash C}\ \mathrm{DE} \quad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \mathrm{CI}$$

$$\frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \mathrm{CEL} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \mathrm{CER}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash \mathsf{K}\, A}\ \mathrm{KR} \qquad \frac{\Gamma \vdash \mathsf{K}\,(A \supset B)}{\Gamma \vdash \mathsf{K}\, A \supset \mathsf{K}\, B}\ \mathrm{KD} \qquad \frac{\Gamma \vdash \mathsf{K}\, A}{\Gamma \vdash \neg\neg A}\ \mathrm{KF}$$

Quasi-Completeness implies WDNS

Proof outline:

1. Assume $\neg(\forall n.\, \neg p\, n \vee \neg\neg p\, n)$ for a contradiction
2. Consider the theory $\mathcal{T} := \{x_n \vee \neg x_n\} \cup \{x_n \mid p\, n\} \cup \{\neg x_n \mid \neg p\, n\}$
3. Observe $\mathcal{T} \nvdash \bot$, exploiting finitely many case distinctions applicable in the negative goal
4. By Quasi-Completeness $\mathcal{T} \Vdash \bot$ remains to show, so assume $\mathcal{M} \Vdash \mathcal{T}$ for a contradiction
5. We now show $\forall n.\, \neg p\, n \vee \neg\neg p\, n$, so assume a particular n
6. By $\mathcal{M} \Vdash \mathcal{T}$ we have $\mathcal{M} \Vdash x_n \vee \neg x_n$, so either $\mathcal{M} \Vdash x_n$ or $\mathcal{M} \Vdash \neg x_n$
7. Then either $\neg\neg p\, n$ or $\neg p\, n$ must be the case, respectively