Trakhtenbrot's Theorem in Coq

A Constructive Approach to Finite Model Theory

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Fragments of the Entscheidungsproblem

Given a first-order formula φ, is there a model $\mathcal{M}$ with $\mathcal{M} \vDash \varphi$?

Church and Turing: general problem is undecidable[1]

Restrictions on φ:

- Quantifier prefix (complexity of alternations)
- Signature (monadic vs. at least binary symbols)

Restrictions on $\mathcal{M}$:

- Finite satisfiability (FSAT) also undecidable [Trakhtenbrot, 1950]
- Dual to general problem: finite validity not enumerable

[1] Coq mechanisation presented at last year's CPP [Forster et al., 2019]

Goal: Mechanise Trakhtenbrot's Theorem in Coq

Textbook proofs by reduction from the halting problem:[2]

- Encode Turing machine M as formula φ_M over custom signature
- Verify that the models of φ_M correspond to the runs of M
- Conclude that M halts if and only if φ_M has a finite model

Challenges for mechanisation:

- Reducing to binary signature usually left as exercise
- Requires a mechanised model of Turing machines
- Reduction function $M \mapsto \varphi_M$ needs to be shown computable!

[2] cf. [Libkin, 2010, Börger et al., 1997]

Simplification: Synthetic Computability

Every function definable in constructive type theory is computable

$\Downarrow$

Computability theory without a concrete model of computation!

A predicate $p : X \to \mathbb{P}$...

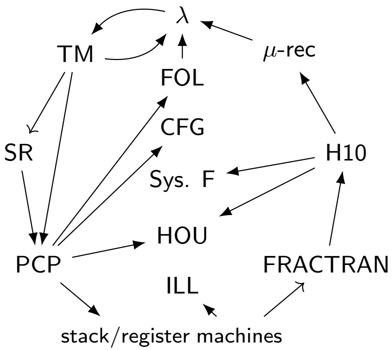
- is decidable: $\exists f : X \to \mathbb{B}. \forall x. f\, x = \mathsf{tt} \leftrightarrow p\, x$
- is enumerable: $\exists g : \mathbb{N} \to X. \forall x. \exists n. g\, n = x$
- reduces to $q : Y \to \mathbb{P}$: $\exists h : X \to Y. \forall x. p\, x \leftrightarrow q\, (h\, x)$

$\Rightarrow$ No need to encode f, g, and h as Turing machines!

Where to start?

But don't we need Turing machines to base the reduction to FSAT?

We can pick any problem from our Coq library of undecidability proofs:



Post correspondence problem (PCP):

- Domino-like matching problem
- Only involves boolean strings
- Easy to express in Coq's type theory
- Easy to encode in first-order logic
- Reduction PCP $\preceq$ FSAT easy to verify

Challenges: Constructive Finite Model Theory

Model Theory $\subseteq$ Set Theory $\cup$ Classical Logic

Working in constructive type theory poses a few questions:

- What is a a good rendering of models?
- What is a good rendering of finiteness?
- Do the tools from model theory transport to our setting?
- Which other tools do we need to invent?

First-Order Satisfiability

Given a signature $\Sigma = (\mathcal{F}_\Sigma; \mathcal{P}_\Sigma)$, we represent terms and formulas by:

$$t \;:\; \mathsf{Term}_\Sigma \;::=\; x \mid f\,\vec{t} \qquad\qquad (x : \mathbb{N},\ f : \mathcal{F}_\Sigma,\ \vec{t} : \mathsf{Term}_\Sigma^{|f|})$$

$$\varphi, \psi \;:\; \mathsf{Form}_\Sigma \;::=\; \dot\bot \mid P\,\vec{t} \mid \varphi \,\dot\Box\, \psi \mid \dot\nabla \varphi \qquad\qquad (P : \mathcal{P}_\Sigma,\ \vec{t} : \mathsf{Term}_\Sigma^{|P|})$$

A (Tarski) model $\mathcal{M}$ over a domain D is a pair of interpretation functions:

$$\_^{\mathcal{M}} \;:\; \forall f : \mathcal{F}_\Sigma.\, D^{|f|} \to D \qquad\qquad \_^{\mathcal{M}} \;:\; \forall P : \mathcal{P}_\Sigma.\, D^{|P|} \to \mathbb{P}$$

For assignments $\rho : \mathbb{N} \to D$ define evaluation $\hat\rho\, t$ and satisfaction $\mathcal{M} \vDash_\rho \varphi$:

$$\hat\rho\, x \;:=\; \rho x \qquad\qquad \hat\rho\,(f\,\vec{t}) \;:=\; f^{\mathcal{M}}\,(\hat\rho\,\vec{t})$$

$$\mathcal{M} \vDash_\rho \dot\bot \;:=\; \bot \qquad\qquad \mathcal{M} \vDash_\rho \varphi \,\dot\Box\, \psi \;:=\; \mathcal{M} \vDash_\rho \varphi \,\Box\, \mathcal{M} \vDash_\rho \psi$$

$$\mathcal{M} \vDash_\rho P\,\vec{t} \;:=\; P^{\mathcal{M}}\,(\hat\rho\,\vec{t}) \qquad\qquad \mathcal{M} \vDash_\rho \dot\nabla \varphi \;:=\; \nabla a : D.\, \mathcal{M} \vDash_{a \cdot \rho} \varphi$$

$\mathsf{SAT}(\Sigma)\,\varphi$:= there are $\mathcal{M}$ and ρ such that $\mathcal{M} \vDash_\rho \varphi$

Finiteness in Constructive Type Theory

Definition

A type X is finite if there exists a list l_X with $x \in l_X$ for all $x : X$.

This seems to be a good compromise:

- Easy to establish and work with
- Does not enforce discreteness (decidable equality)
- Enough to get expected properties:
  - Every strict order on a finite type is well-founded
  - Every finite decidable equivalence relation admits a quotient on $\mathbb{F}_n$

$\mathsf{FSAT}(\Sigma)\,\varphi$ if additionally D is finite and all $P^{\mathcal{M}}$ are decidable

$\mathsf{FSATEQ}(\Sigma;\equiv)\,\varphi$ if $x \equiv^{\mathcal{M}} y \leftrightarrow x = y$ for all $x, y : D$ (hence discrete)

Encoding the Post Correspondence Problem

Inductive characterisation of PCP on sets R of cards (s, t) of strings:

$$\frac{(s,t) \in R}{R \triangleright (s,t)} \qquad \frac{(s,t) \in R \quad R \triangleright (u,v)}{R \triangleright (s \mathbin{+\!\!+} u, t \mathbin{+\!\!+} v)} \qquad \frac{R \triangleright (s,s)}{\mathrm{PCP}\ R}$$

We use the signature $\Sigma_{\mathrm{BPCP}} := (\{\star^0, e^0, f_{\mathrm{tt}}^1, f_{\mathrm{ff}}^1\}; \{P^2, \prec^2, \equiv^2\})$:

- Chains like $f_{\mathrm{ff}}(f_{\mathrm{tt}}(e))$ represent strings while $\star$ signals overflow
- P concerns only defined values and $\prec$ is a strict ordering:

$$\begin{aligned} \varphi_P &:= \dot\forall xy.\, P\,x\,y \mathrel{\dot\to} x \not\equiv \star \mathrel{\dot\wedge} y \not\equiv \star \\ \varphi_\prec &:= (\dot\forall x.\, x \not\prec x) \mathrel{\dot\wedge} (\dot\forall xyz.\, x \prec y \mathrel{\dot\to} y \prec z \mathrel{\dot\to} x \prec z) \end{aligned}$$

- Sanity checks on f regarding overflow, disjointness, and injectivity:

$$\varphi_f := \begin{pmatrix} f_{\mathrm{tt}}\,\star \equiv \star \mathrel{\dot\wedge} f_{\mathrm{ff}}\,\star \equiv \star \\ \dot\forall x.\, f_{\mathrm{tt}}\,x \not\equiv e \\ \dot\forall x.\, f_{\mathrm{ff}}\,x \not\equiv e \end{pmatrix} \mathrel{\dot\wedge} \begin{pmatrix} \dot\forall xy.\, f_{\mathrm{tt}}\,x \not\equiv \star \mathrel{\dot\to} f_{\mathrm{tt}}\,x \equiv f_{\mathrm{tt}}\,y \mathrel{\dot\to} x \equiv y \\ \dot\forall xy.\, f_{\mathrm{ff}}\,x \not\equiv \star \mathrel{\dot\to} f_{\mathrm{ff}}\,x \equiv f_{\mathrm{ff}}\,y \mathrel{\dot\to} x \equiv y \\ \dot\forall xy.\, f_{\mathrm{tt}}\,x \equiv f_{\mathrm{ff}}\,y \mathrel{\dot\to} f_{\mathrm{tt}}\,x \equiv \star \mathrel{\dot\wedge} f_{\mathrm{ff}}\,y \equiv \star \end{pmatrix}$$

Trakhtenbrot's Theorem

Given an instance R of PCP, we construct a formula φ_R by setting

$$\varphi_R := \varphi_P \mathbin{\dot\land} \varphi_\prec \mathbin{\dot\land} \varphi_f \mathbin{\dot\land} \varphi_\triangleright \mathbin{\dot\land} \dot\exists x.\, P\, x\, x.$$

Crucially, we enforce that P satisfies the inversion principle of $R \triangleright (s,t)$:

$$\varphi_\triangleright := \dot\forall xy.\, P\, x\, y \mathbin{\dot\to} \dot{\bigvee_{(s,t)\in R}} \mathbin{\dot\lor} \begin{cases} x \equiv \overline{s} \mathbin{\dot\land} y \equiv \overline{t} \\ \dot\exists uv.\, P\, u\, v \mathbin{\dot\land} x \equiv \overline{s} \mathbin{+\!\!\!+\!\!\!+} u \mathbin{\dot\land} y \equiv \overline{t} \mathbin{+\!\!\!+\!\!\!+} v \mathbin{\dot\land} u/v \prec x/y \end{cases}$$

Theorem

PCP R *iff* $\mathrm{FSATEQ}(\Sigma_{\mathrm{BPCP}};\equiv)\varphi_R$, *hence* PCP $\preceq \mathrm{FSATEQ}(\Sigma_{\mathrm{BPCP}};\equiv)$.

Proof.

If R has a solution of length n, then φ_R is satisfied by the model of strings of length bounded by n. Conversely, if $\mathcal{M} \vDash_\rho \varphi_R$ we can extract a solution of R from $\varphi_\triangleright$ by well-founded induction on $\prec^{\mathcal{M}}$ (which is applicable since $\mathcal{M}$ is finite). $\square$

Signature Transformations

Given a finite and discrete signature Σ with arities bounded by n, we have:

$$\mathsf{FSATEQ}(\Sigma; \equiv) \preceq \mathsf{FSAT}(\Sigma) \preceq \mathsf{FSAT}(\mathbb{0}; P^{n+2}) \preceq \mathsf{FSAT}(\mathbb{0}; \in^2)$$

First reduction: axiomatise that $\equiv$ is a congruence for the symbols in Σ

Second reduction:

- Encode k-ary functions as $(k+1)$-ary relations
- Align the relation arities to be constantly $n+1$
- Merge relations into a single $(n+2)$-ary relation indexed by constants
- Interpret constants with fresh variables

Caveat: intermediate reductions may rely on discrete models...

Discrete Models

$\mathsf{FSAT}'(\Sigma)\,\varphi$ if $\mathsf{FSAT}(\Sigma)\,\varphi$ on a discrete model

Can every finite model $\mathcal{M}$ be transformed to a discrete finite model $\mathcal{M}'$?

Idea: first-order indistinguishability $x \doteq y \;:=\; \forall \varphi \rho.\, \mathcal{M} \vDash_{x\cdot\rho} \varphi \leftrightarrow \mathcal{M} \vDash_{y\cdot\rho} \varphi$

Lemma

The relation $x \doteq y$ is a decidable congruence for the symbols in Σ.

Fact

$\mathsf{FSAT}'(\Sigma)\,\varphi$ *iff* $\mathsf{FSAT}(\Sigma)\,\varphi$, *hence in particular* $\mathsf{FSAT}'(\Sigma)\,\varphi \preceq \mathsf{FSAT}(\Sigma)\,\varphi$.

Proof.

If $\mathcal{M} \vDash_\rho \varphi$ pick $\mathcal{M}'$ to be the quotient of $\mathcal{M}$ under $x \doteq y$. $\square$

Compressing Relations: $\mathsf{FSAT}(\mathbb{0}; P^n) \preceq \mathsf{FSAT}(\mathbb{0}; \in^2)$

Intuition: encode $P\,x_1 \ldots x_n$ as $(x_1, \ldots, x_n) \in p$ for a set p representing P

So let's play set theory! For a set d representing the domain we define $\varphi'_\in$:

$$\begin{aligned} (P\,x_1 \ldots x_n)'_\in &:= \text{“}(x_1, \ldots, x_n) \in p\text{”} & (\dot\forall z.\,\varphi)'_\in &:= \dot\forall z.\, z \in d \mathrel{\dot\to} (\varphi)'_\in \\ (\varphi \mathbin{\dot\square} \psi)'_\in &:= (\varphi)'_\in \mathbin{\dot\square} (\psi)'_\in & (\dot\exists z.\,\varphi)'_\in &:= \dot\exists z.\, z \in d \mathbin{\dot\wedge} (\varphi)'_\in \end{aligned}$$

Then $\varphi_\in$ is $\varphi'_\in$ plus asserting $\in$ to be extensional and d to be non-empty.

Fact

$\mathsf{FSAT}(\mathbb{0}; P^n)\,\varphi$ *iff* $\mathsf{FSAT}(\mathbb{0}; \in^2)\,\varphi_\in$, *hence* $\mathsf{FSAT}(\mathbb{0}; P^n) \preceq \mathsf{FSAT}(\mathbb{0}; \in^2)$.

Proof.

The hard direction is to construct a model of $\varphi_\in$ given a model $\mathcal{M}$ of φ. We employ a segment of the model of hereditarily finite sets by [Smolka and Stark, 2016] large enough to accommodate $\mathcal{M}$. □

Full Signature Classification

Composing all signature transformations verified in the paper we obtain:

Theorem

If Σ contains either an at least binary relation or a unary relation together with an at least binary function, then PCP *reduces to* $\mathsf{FSAT}(\Sigma)$.

On the other hand, FSAT for monadic signatures remains decidable:

Theorem

If Σ is discrete and has all arities bounded by 1 *or if all relation symbols have arity* 0*, then* $\mathsf{FSAT}(\Sigma)$ *is decidable.*

In any case, since one can enumerate all finite models up to extensionality:

Fact

If Σ is discrete and enumerable, then $\mathsf{FSAT}(\Sigma)$ *is enumerable.*

Coq Mechanisation[4]

- Includes all results presented in the paper (PDF is hyperlinked!)
- Roughly 10k loc with additional 3k loc of utility libraries
  - More than 4k loc for $\mathsf{FSAT}(\mathbb{0}; P^n) \preceq \mathsf{FSAT}(\mathbb{0}; \in^2)$
  - Less than 500 loc for $\mathsf{PCP} \preceq \mathsf{FSATEQ}(\Sigma_{\mathsf{BPCP}}; \equiv)$
- FOL engineering similar to previous devs (cf. [Forster et al., 2020])
  - De Bruijn encoding of bound variables
  - Dependent syntax enforcing well-defined terms and formulas
- Axiom-free to ensure computability and interoperability
- Contributed to the Coq library of undecidability proofs[3]

[3] https://github.com/uds-psl/coq-library-undecidability

[4] https://www.ps.uni-saarland.de/extras/fol-trakh/

Future Work

- Generalise some intermediate transformations from FSAT to SAT
- Explore and mechanise some direct consequences:
  - Undecidability of query containment and equivalence on data bases
  - Undecidability of separation logic
- Study further undecidability results in first-order logic
  - Deduction in axiom systems like Peano arithmetic or ZF set theory
  - Strengthen the result given in [Forster et al., 2019] to binary signatures
- Mechanise the classification of satisfiability wrt. quantifier prefixes

What to take home?

- Synthetic computability simplifies undecidability proofs
- PCP is a good starting point for reductions into FOL
- Initial reduction was easy, signature transformations were tough
- Constructive (finite) model theory has interesting structure to explore
- Undecidability library open for contributions!

Thank You!

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Disclaimer: Synthetic Undecidability

Coq is consistent with axioms making all problems decidable!

- FSAT not shown undecidable in the sense that there is no Coq decider
- Only reductions like PCP $\preceq$ FSAT can be verified
- Axioms like undecidability of PCP then have expected consequences

Alternative: switch to a concrete model of computation

- Pro: problems can be shown to have no deciding e.g. Turing machine
- Contra: reductions need to be encoded as e.g. Turing machines
- Tool support in principle possible (cf. [Forster and Kunze, 2019])

First-Order Indistinguishability

We define operators $\mathrm{F}_{\mathcal{F}}, \mathrm{F}_{\mathcal{P}} : (D \to D \to \mathbb{P}) \to (D \to D \to \mathbb{P})$ by :

$$\mathrm{F}_{\mathcal{F}}(\mathcal{R})\, x\, y := \forall f.\, f \in l_{\mathcal{F}} \to \forall (\vec{v} : D^{|f|})\, (i : \mathbb{F}_{|f|}).\, \mathcal{R}\left(f^{\mathcal{M}}\, \vec{v}[x/i]\right) \left(f^{\mathcal{M}}\, \vec{v}[y/i]\right)$$
$$\mathrm{F}_{\mathcal{P}}(\mathcal{R})\, x\, y := \forall P.\, P \in l_{\mathcal{P}} \to \forall (\vec{v} : D^{|P|})\, (i : \mathbb{F}_{|P|}).\, P^{\mathcal{M}}\, \vec{v}[x/i] \leftrightarrow P^{\mathcal{M}}\, \vec{v}[y/i]$$

We then consider $\mathrm{F}(\mathcal{R}) := \mathrm{F}_{\mathcal{F}}(\mathcal{R}) \cap \mathrm{F}_{\mathcal{P}}(\mathcal{R})$ and show:

Theorem

First-order indistinguishability $\doteq$ up to $l_{\mathcal{F}}/l_{\mathcal{P}}$ is extensionally equivalent to $\equiv_{\mathrm{F}}$ (Kleene's greatest fixpoint of F), i.e. for any $x, y : D$ we have

$$x \doteq y \;\leftrightarrow\; x \equiv_{\mathrm{F}} y \quad \textit{where} \quad x \equiv_{\mathrm{F}} y \;:=\; \forall n : \mathbb{N}.\, \mathrm{F}^n(\lambda uv.\top)\, x\, y.$$

Moreover, the relation $x \equiv_{\mathrm{F}} y$ is decidable and hence so is $x \doteq y$.

Hereditarily Finite Sets

Theorem

Given a decidable n-ary relation $R : X^n \to \mathbb{P}$ *over a finite, discrete and inhabited type* X*, one can compute a finite and discrete type* Y *equipped with a decidable relation* $\in : Y \to Y \to \mathbb{P}$*, two distinguished elements* $d, r : Y$ *and a pair of maps* $i : X \to Y$ *and* $s : Y \to X$ *s.t.*

1. $\in$ *is extensional;*
2. *extensionally equal elements of* Y *are equal;*
3. *all n-tuples of members of* d *exist in* Y*;*
4. $\forall x : X.\, i\,x \in d$*;*
5. $\forall y : Y.\, y \in d \to \exists x.\, y = i\,x$*;*
6. $\forall x : X.\, s(i\,x) = x$*;*
7. $R\,\vec{v}$ *iff* $i(\vec{v})$ *is a n-tuple member of* r*, for any* $\vec{v} : X^n$*.*

Proof.

The type Y is built from the type of hereditarily finite sets. The idea is first to construct d as a transitive set of which the elements are in bijection i/s with the type X, hence d is the cardinal of X in the set-theoretic meaning. Then the iterated powersets $\mathcal{P}(d), \mathcal{P}^2(d), \ldots, \mathcal{P}^k(d)$ are all transitive as well and contain d both as a member and as a subset. Considering $\mathcal{P}^{2n}(d)$ which contains all the n-tuples built from the members of d, we define r as the set of n-tuples collecting the encoding $i(\vec{v})$ of vectors $\vec{v} : X^n$ such that $R\,\vec{v}$. We show $r \in p$ for p defined as $p := \mathcal{P}^{2n+1}(d)$. Then we define $Y := \{z \mid z \in p\}$ and restrict membership $\in$ to Y. □

Decidability Results

Lemmas used for decidability of monadic FOL and enumerability of FSAT:

Lemma

Given a discrete signature Σ and a discrete and finite type D, one can decide whether or not a formula over Σ has a (finite) model over D.

Lemma

A formula over a signature Σ has a finite and discrete model if and only if it has a (finite) model over $\mathbb{F}_n$ for some $n : \mathbb{N}$.