Completeness Theorems for First-Order Logic Analysed in Constructive Type Theory

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Is the completeness theorem of FOL algorithmic?

Gödel 1930, Henkin 1949

Classical completeness proofs

Kreisel 1962

Standard completeness requires Markov's principle

Veldman 1976

Modified semantics yield fully constructive completeness

Herbelin/Ilik 2016

Computational analysis in constructive type theory

Constructive type theory as a formal meta-logic

Reification

Deductions $\vdash \varphi$ can be extracted from meta-level proofs of $\vDash \varphi$

Conservativity

Meta-logic and formal object-logic "agree" on first-order formulas

Admissibility

Particular instances of completeness may be provable

Computability

Internal notion of computation allows direct analysis

Main Contributions

Constructive analysis of completeness theorems:

- Model-theoretic semantics: uniform analysis of standard, exploding, and minimal semantics of the $\rightarrow, \forall, \bot$-fragment; admissibility results
- Algebraic semantics: constructive completeness proofs for algebras with explicit atom interpretations based on Scott '08
- Game semantics: instantiation of a general isomorphism[1] between winning strategies and deductions to intuitionistic first-order logic; streamlined representation of dialogues as state transition systems

Reusable Coq library for first-order logic hyperlinked with the PDF:
`https://www.ps.uni-saarland.de/extras/fol-completeness/`

[1] Sørensen and Urzyczyn '08

Constructive Type Theory

Constructive Type Theory

Features as implemented in Coq's type theory:

- Inductive types: $\mathbb{B}$, $\mathbb{N}$, lists $\mathcal{L}(X)$, vectors X^n, ...
- Standard type formers: $X \to Y$, $X \times Y$, $X + Y$, $\forall x.\, F\,x$, $\Sigma x.\, F\,x$
- Propositional universe $\mathbb{P}$ with logical connectives: $\to$, $\wedge$, $\vee$, $\forall$, $\exists$

All definable functions are computable!

So standard notions from computability theory can be synthesised:

Definition

Let X be a type and $p : X \to \mathbb{P}$ be a predicate. We say that p is

- decidable if there is $f : X \to \mathbb{B}$ with $\forall x.\, p\,x \leftrightarrow f\,x = \mathrm{tt}$
- enumerable if there is $f : \mathbb{N} \to X$ with $\forall x.\, p\,x \leftrightarrow \exists n.\, f\,n = \ulcorner x \urcorner$

Markov's Principle: 2 Versions

"Termination is stable (under double negation)"

For the computation internal to constructive type theory:

$$\mathsf{MP} := \forall f : \mathbb{N} \to \mathbb{B}.\, \neg\neg(\exists n.\, f\; n = \mathsf{tt}) \to \exists n.\, f\; n = \mathsf{tt}$$

For the cbv. lambda calculus L as a formal model of computation:[2]

$$\mathsf{MP_L} := \forall s : \mathsf{L}.\, \neg\neg\mathcal{E}\; s \to \mathcal{E}\; s \qquad (\mathcal{E}\; s \;:=\; \text{"}s \text{ terminates"})$$

- MP implies $\mathsf{MP_L}$ (since $\mathcal{E}$ is enumerable)
- MP is independent but admissible in Coq's type theory[3]
- $\mathsf{MP_L}$ is independent but admissible in Coq's type theory

[2] Plotkin '75, Forster/Smolka '17

[3] Coquand/Mannaa '17, Pédrot/Tabareau '18

First-order Logic

Syntax

Represented as inductive type over signature Σ providing function symbols $f : \mathcal{F}_\Sigma$ and relation symbols $P : \mathcal{P}_\Sigma$ with arities $|f|$ and $|P|$:

$$t : \mathbb{T} ::= x \mid f\,\vec{t} \qquad (x : \mathbb{N})$$

$$\varphi, \psi : \mathbb{F} ::= \dot{\bot} \mid P\,\vec{t} \mid \varphi \dot{\to} \psi \mid \varphi \dot{\land} \psi \mid \varphi \dot{\lor} \psi \mid \dot{\forall} \varphi \mid \dot{\exists} \varphi$$

We set $\dot{\neg}\varphi := \varphi \dot{\to} \dot{\bot}$ and denote the $\to, \forall, \bot$-fragment by $\mathbb{F}^*$.

De Bruijn encoding of quantifiers to avoid naming conflicts:

$$\text{“}P\,x\,y \to \forall x.\, \exists y.\, P\,x\,y\text{”} \qquad \rightsquigarrow \qquad P\,7\,4 \dot{\to} \dot{\forall}\,\dot{\exists} P\,1\,0$$

The variables 7 and 4 in this example are free and variables that do not occur freely are fresh. A formula with no free variables is closed.

Deduction Systems

Represented as inductive predicates of the form $\mathcal{L}(\mathbb{F}) \to \mathbb{F} \to \mathbb{P}$:

$$\frac{\uparrow\Gamma \vdash \varphi}{\Gamma \vdash \dot\forall\varphi}\ \mathsf{AI} \qquad \frac{\Gamma \vdash \dot\forall\varphi}{\Gamma \vdash \varphi[t]}\ \mathsf{AE}$$

$$\frac{\Gamma \vdash \varphi[t]}{\Gamma \vdash \dot\exists\varphi}\ \mathsf{EI} \qquad \frac{\Gamma \vdash \dot\exists\varphi \quad \uparrow\Gamma, \varphi \vdash \uparrow\psi}{\Gamma \vdash \psi}\ \mathsf{EE}$$

$$\dots$$

Given Γ, φ, and ψ one can compute a fresh variable x such that

1. $\uparrow\Gamma \vdash \varphi$ iff $\Gamma \vdash \varphi[x]$ and
2. $\uparrow\Gamma, \varphi \vdash \uparrow\psi$ iff $\Gamma, \varphi[x] \vdash \psi$.

Classical variant $\Gamma \vdash_c \varphi$ obtained by adding $\Gamma \vdash_c ((\varphi \dot\to \psi) \dot\to \varphi) \dot\to \varphi$.
Natural generalisation to $\mathcal{T} \vdash \varphi$ for theories $\mathcal{T} : \mathbb{F} \to \mathbb{P}$.

Computational Properties

If Σ is a data type (enumerable and discrete), then:

- $\mathbb{T}$ and $\mathbb{F}$ are data types
- $\Gamma \vdash_c \varphi$ is enumerable
- $\mathsf{MP_L}$ is equivalent to the stability of $\Gamma \vdash_c \varphi$
- Stability of $\Gamma \vdash_c \varphi$ is independent but admissible
- $\mathcal{T} \vdash_c \varphi$ for enumerable $\mathcal{T}$ behaves similarly wrt. MP

Strategic consequences:

1. Analysing completeness up to the stability of deduction suffices
2. Completeness can be analysed on two levels (MP and $\mathsf{MP_L}$)
3. Some formulations of completeness may be admissible

Model-Theoretic Semantics

(Considering enumerable Σ and the $\mathbb{F}^*$-fragment)

Tarski Semantics

A (Tarski) model $\mathcal{M}$ over a domain D is a family of functions

$$f^{\mathcal{M}} \; : \; D^{|f|} \to D \qquad\qquad P^{\mathcal{M}} \; : \; D^{|P|} \to \mathbb{P}.$$

Assignments $\rho : \mathbb{N} \to D$ are extended to term evaluations $\hat{\rho} : \mathbb{T} \to D$.
The relation $\mathcal{M} \vDash_\rho$ for formulas $\varphi : \mathbb{F}^*$ is defined recursively by:

$$\mathcal{M} \vDash_\rho \dot{\bot} \;:=\; \bot \qquad\qquad \mathcal{M} \vDash_\rho \varphi \dot{\to} \psi \;:=\; \mathcal{M} \vDash_\rho \varphi \to \mathcal{M} \vDash_\rho \psi$$

$$\mathcal{M} \vDash_\rho P\,\vec{t} \;:=\; P^{\mathcal{M}}\,(\hat{\rho}\,\vec{t}\,) \qquad\qquad \mathcal{M} \vDash_\rho \dot{\forall}\,\varphi \;:=\; \forall a : D.\, \mathcal{M} \vDash_{a;\rho} \varphi$$

$\mathcal{M}$ is called classical if $\mathcal{M} \vDash ((\varphi \dot{\to} \psi) \dot{\to} \varphi) \dot{\to} \varphi$ for all $\varphi, \psi : \mathbb{F}^*$.
We write $\mathcal{T} \vDash \varphi$ if $\mathcal{M} \vDash_\rho \varphi$ for every classical $\mathcal{M}$ and ρ with $\mathcal{M} \vDash_\rho \mathcal{T}$.

Tarski Semantics: Standard Completeness

The central model existence theorem has a constructive proof:[4]

Theorem

Every consistent (and closed) theory is satisfied in a classical model.

Model existence yields quasi-completeness: $\mathcal{T} \vDash \varphi$ implies $\neg\neg(\mathcal{T} \vdash_c \varphi)$
⇒ Single applications of MP and $\mathsf{MP_L}$ yield completeness statements
⇒ These completeness statements are admissible

Leads to the following characterisations:

Theorem

- *Completeness of* $\Gamma \vdash_c \varphi$ *is equivalent to* $\mathsf{MP_L}$.
- *Completeness of* $\mathcal{T} \vdash_c \varphi$ *for enumerable* $\mathcal{T}$ *is equivalent to* MP.
- *Completeness of* $\mathcal{T} \vdash_c \varphi$ *for arbitrary* $\mathcal{T}$ *is equivalent to* EM.

[4] Herbelin/Ilik '16

Tarski Semantics: Constructive Completeness

Generalise semantics to admit exploding models:[5]

- Extend models with falsity interpretation $\bot^{\mathcal{M}} : \mathbb{P}$
- $\mathcal{M}$ is exploding if $\mathcal{M} \vDash \dot{\bot} \dot{\rightarrow} \varphi$ for all $\varphi : \mathbb{F}^*$
- $\mathcal{T} \vDash_e \varphi$ if $\mathcal{M} \vDash_\rho \varphi$ for all exploding classical $\mathcal{M}$ and ρ with $\mathcal{M} \vDash_\rho \mathcal{T}$

Generalised model existence yields constructive completeness:

Theorem

For every (closed) theory $\mathcal{T}$ there is an exploding classical model $\mathcal{M}$ and an assignment ρ such that (1) $\mathcal{M} \vDash_\rho \mathcal{T}$ and (2) $\mathcal{M} \vDash_\rho \dot{\bot}$ implies $\mathcal{T} \vdash_c \dot{\bot}$.

Corollary

$\mathcal{T} \vDash_e \varphi$ implies $\mathcal{T} \vdash_c \varphi$ (for closed $\mathcal{T}$ and φ).

[5] Veldman '76

Kripke Semantics (cf. Herbelin/Lee '09)

Analogous properties hold for Kripke entailment $\Gamma \Vdash \varphi$.

Universal Kripke models yield completeness as follows:

- $\Gamma \Vdash_e \varphi$ implies $\Gamma \vdash \varphi$
- $\Gamma \Vdash \varphi$ implies $\neg\neg(\Gamma \vdash \varphi)$

These in fact hold for a cut-free sequent calculus $\Gamma \Rightarrow \varphi$, therefore establishing a normalisation-by-evaluation procedure.

Again, the stability assumptions are necessary, so we conclude:

Theorem

- *Completeness of* $\Gamma \vdash \varphi$ *is equivalent to* $\mathsf{MP_L}$*.*
- *Completeness of* $\mathcal{T} \vdash \varphi$ *for enumerable* $\mathcal{T}$ *implies* MP*.*
- *Completeness of* $\mathcal{T} \vdash \varphi$ *for arbitrary* $\mathcal{T}$ *implies* EM*.*

Algebraic Semantics

Algebraic Semantics (cf. Scott '08)

Given a complete Heyting algebra $(\mathcal{H}, \leq, 0, \bigsqcap, \bigsqcup, \Rightarrow)$ we extend atom interpretations $[\![P\,\vec{t}\,]\!] : \mathcal{H}$ to all formulas in $\mathbb{F}$:

$$[\![\dot{\bot}]\!] := 0 \qquad [\![\varphi \dot{\wedge} \psi]\!] := [\![\varphi]\!] \sqcap [\![\psi]\!] \qquad [\![\dot{\forall}\varphi]\!] := \bigsqcap_t [\![\varphi[t]]\!]$$

$$[\![\varphi \dot{\rightarrow} \psi]\!] := [\![\varphi]\!] \Rightarrow [\![\psi]\!] \qquad [\![\varphi \dot{\vee} \psi]\!] := [\![\varphi]\!] \sqcup [\![\psi]\!] \qquad [\![\dot{\exists}\varphi]\!] := \bigsqcup_t [\![\varphi[t]]\!]$$

A formula φ is valid whenever $x \leq [\![\varphi]\!]$ for all x in every $\mathcal{H}$.

The Lindenbaum algebra $\mathcal{L} = (\mathbb{F}, \varphi \vdash \psi, \dot{\bot}, \dot{\wedge}, \dot{\vee}, \dot{\rightarrow})$ can be completed via a standard construction which then witnesses $[\![\varphi]\!] \equiv \lambda\psi.\, \psi \vdash \varphi$, so:

Theorem

If φ is valid in every complete Heyting algebra, then $\vdash \varphi$.

This generalises to $\Gamma \vdash_c \varphi$ and complete Boolean algebras.

Game Semantics

Game Semantics (cf. Sørensen/Urzyczyn '08)

We represent intuitionistic E-dialogues parametrised over $(\mathbb{F}', \mathbb{F}^0, \mathcal{A}, \triangleright, \mathcal{D})$.
The attacks and defenses for the concrete instance of FOL are as follows:

$$a_{\dot\bot} \triangleright \dot\bot \qquad \mathcal{D}_{\dot\bot} = \{\} \qquad a_{\dot\to} \mid \varphi \triangleright \varphi \dot\to \psi \qquad \mathcal{D}_{\dot\to} = \{\psi\}$$

$$a_t \triangleright \dot\forall \varphi \qquad \mathcal{D}_t = \{\varphi[t]\} \qquad a_{\dot\vee} \triangleright \varphi \dot\vee \psi \qquad \mathcal{D}_{\dot\vee} = \{\varphi, \psi\}$$

$$a_{\dot\exists} \triangleright \dot\exists \varphi \qquad \mathcal{D}_{\dot\exists} = \{\varphi[t] \mid t : \mathbb{T}\} \qquad a_L \triangleright \varphi \dot\wedge \psi \qquad \mathcal{D}_L = \{\varphi\}$$

- The player and opponent take turns in manipulating the game state (A, a) containing the opponent's current admissions A and attack a
- The player may defend against a or attack any formular from A
- The opponent reacts by attacking the player's defense or by defending/countering against the player's attack
- A state is winning if for all allowed player moves every possible opponent reaction leads to a winning state
- A formula is E-valid if $([\varphi], a)$ is winning for all initial attacks a on φ

Game Semantics (cf. Sørensen and Urzyczyn '08)

A sequent calculus LJD of type $\mathcal{L}(\mathbb{F}') \to (\mathbb{F}' \to \mathbb{P}) \to \mathbb{P}$ is defined by:

$$\frac{\varphi \in \mathcal{S} \quad \text{justified } \Gamma\, \varphi \quad \forall a \,|\, \psi \rhd \varphi.\ \Gamma, \psi \Rightarrow_D \mathcal{D}_a}{\Gamma \Rightarrow_D \mathcal{S}}\ \mathsf{R}$$

$$\frac{\varphi \in \Gamma \quad \text{justified } \Gamma\, \psi \quad a \,|\, \psi \rhd \varphi \quad \forall\, \theta \in \mathcal{D}_a.\ \Gamma, \theta \Rightarrow_D \mathcal{S} \quad \forall a' \,|\, \tau \rhd \psi.\ \Gamma, \tau \Rightarrow_D \mathcal{D}_{a'}}{\Gamma \Rightarrow_D \mathcal{S}}\ \mathsf{L}$$

Winning strategies and LJD derivations are easily shown isomorphic, so:

Theorem

Any formula φ is E-valid if and only if one can derive $[] \Rightarrow_D \{\varphi\}$.

The concrete instance of LJD for FOL is equivalent to the standard sequent calculus LJ defined by judgements $\Gamma \Rightarrow_J \varphi$:

Theorem

Any formula φ is E-valid if and only if one can derive $[] \Rightarrow_J \varphi$.

Discussion

What else is in the paper?

- Constructive completeness proofs for minimal logic ($\rightarrow, \forall$-fragment)
  - by a generalised model existence theorem for Tarski semantics
  - by the same exploding universal model for Kripke semantics

- Completeness of $\mathcal{T} \vdash_c \varphi$ for formulas with free variables

- Independence of $\mathsf{MP_L}$ as special case of Pédrot/Tabareau '18: $\mathsf{MP_L}$ with independence of premise rule IP yields a decider for $\mathcal{E}$

- Reductions establishing the equivalence of $\mathsf{MP_L}$ and stability of ND:
  - $\mathcal{E}$ reduces to $\vdash_c$ (chaining reductions from previous work[6])
  - $\vdash$ reduces to $\mathcal{E}$ (by showing that $\vdash$ is L-enumerable[7])
  - $\vdash_c$ reduces to $\vdash$ (double negation translation)

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[6] Wuttke '18, Forster et al. '18, and Forster et al. '19.

[7] Using the extraction framework from Forster/Kunze '19.

Coq Formalisation

`https://www.ps.uni-saarland.de/extras/fol-completeness/`

- About 7500 lines of code
- All main results parametric in the signature Σ
- De Bruijn encoding of binders supported by Autosubst[8]
- Usage of type classes, parametric deduction systems, Equations package, special ND tactics, etc. to ease mechanisation
- Part of a growing Coq library of undecidability proofs[9]

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[8] `https://www.ps.uni-saarland.de/~kstark/autosubst2/`
[9] `https://github.com/uds-psl/coq-library-undecidability`

Future Work

Extend the completeness library:

- Model-theoretic semantics including $\vee$ and $\exists$
- Strong completeness ($\mathcal{T} \vDash \varphi \;\rightarrow\; \mathcal{T} \vdash \varphi$) for more semantics
- Hybrid semantics such as mix of model-theoretic and algebraic parts
- Second-order and higher-order logic

Other related ideas:

- Analyse the (upward) Löwenheim-Skolem theorem
- Study first-order axiom systems (undecidability, incompleteness, ...)
- Implement a proof extraction procedure
- Separate MP from $\mathsf{MP_L}$
- Find an effective model with classical reasoning in $\mathbb{P}$

Related Work I

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Wim Veldman.
An intuitiomstic completeness theorem for intuitionistic predicate logic 1.
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PhD thesis, Ecole Polytechnique X, 2010.

Related Work II

Hugo Herbelin and Danko Ilik.
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Draft, 2016.

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Yannick Forster and Gert Smolka.
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Related Work III

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Morten Heine Sørensen and Pawel Urzyczyn.
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Dana Scott.
The algebraic interpretation of quantifiers: Intuitionistic and classical.
In V.W. Marek A. Ehrenfeucht and M. Srebrny, editors, *Andrzej Mostowski and Foundational Studies*. IOS Press, 2008.

Dominik Wehr.
A Constructive Analysis of First-Order Completeness Theorems in Coq, 2019.
Bachelor's thesis, Saarland University.

Maximilian Wuttke.
Verified Programming Of Turing Machines In Coq, 2018.
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Details on the Coq Formalisation

Section	Specification	Proofs
Preliminaries Autosubst	169	53
Preliminaries for $\mathbb{F}^*$	680	599
Tarski Semantics	655	682
Kripke Semantics	342	255
On Markov's Principle	593	978
Preliminaries for $\mathbb{F}$	523	430
Heyting Semantics	297	456
Dialogue Semantics	312	488
Total	3571	3941

Completeness implies stability

Fact

$(\forall \Gamma \varphi.\, \Gamma \vDash \varphi \to \Gamma \vdash_c \varphi) \to \forall \Gamma \varphi.\, (\neg\neg \Gamma \vdash_c \varphi) \to \Gamma \vdash_c \varphi$

Proof.

Assume completeness and let $\neg\neg(\Gamma \vdash_c \varphi)$. To conclude $\Gamma \vdash_c \varphi$ it suffices to show $\Gamma, \dot{\neg}\varphi \vdash_c \dot{\bot}$ and by completeness $\Gamma, \dot{\neg}\varphi \vDash \dot{\bot}$. So for $\mathcal{M} \vDash_\rho \Gamma, \dot{\neg}\varphi$ we have to prove $\bot$, meaning that we may use $\neg\neg(\Gamma \vdash_c \varphi)$ positively as $\Gamma \vdash_c \varphi$. But then $\Gamma \vDash \varphi$ by soundness, contradicting $\mathcal{M} \vDash_\rho \Gamma, \dot{\neg}\varphi$. □

Tarski Semantics: Model Existence

Lemma

For every closed theory $\mathcal{T}$ there is an extension $\mathcal{T}' \supseteq \mathcal{T}$ s.t.:

- *$\mathcal{T}'$ maintains consistency, i.e. $\mathcal{T} \vdash_c \dot{\bot}$ whenever $\mathcal{T}' \vdash_c \dot{\bot}$.*
- *$\mathcal{T}'$ is deductively closed, i.e. $\varphi \in \mathcal{T}'$ whenever $\mathcal{T}' \vdash_c \varphi$.*
- *$\mathcal{T}'$ respects implication, i.e. $\varphi \dot{\to} \psi \in \mathcal{T}'$ iff $\varphi \in \mathcal{T}' \to \psi \in \mathcal{T}'$.*
- *$\mathcal{T}'$ respects universal quantification, i.e. $\dot{\forall}\varphi \in \mathcal{T}'$ iff $\forall t.\, \varphi[t] \in \mathcal{T}'$.*

Extension in two subsequent steps:

1. $\mathcal{H} := \mathcal{T} \cup \{\varphi_n[n] \dot{\to} \dot{\forall} \varphi_n \mid n : \mathbb{N}\}$ is Henkin and maintains consistency
2. $\Omega \supseteq \mathcal{H}$ which is maximal, i.e. $\varphi \in \Omega$ if $\Omega, \varphi \vdash_c \dot{\bot}$ implies $\Omega \vdash_c \dot{\bot}$:

$$\Omega_0 := \mathcal{H} \quad \Omega_{n+1} := \Omega_n \cup \{\varphi_n \mid \Omega_n, \varphi_n \vdash_c \varphi_\bot \to \Omega_n \vdash_c \varphi_\bot\} \quad \Omega := \bigcup_{n:\mathbb{N}} \Omega_n$$

$\Rightarrow$ If $\mathcal{T}$ is consistent, then its extension $\mathcal{T}'$ yields a syntactic model of $\mathcal{T}$.

Tarski Semantics: Model Existence Revisited

Lemma

For every closed formula $\varphi_\bot$ and closed $\mathcal{T}$ there is $\mathcal{T}' \supseteq \mathcal{T}$ s.t.:

- *$\mathcal{T}'$ maintains $\varphi_\bot$-consistency, i.e. $\mathcal{T} \vdash_c \varphi_\bot$ whenever $\mathcal{T}' \vdash_c \varphi_\bot$.*
- *$\mathcal{T}'$ is deductively closed, i.e. $\varphi \in \mathcal{T}'$ whenever $\mathcal{T}' \vdash_c \varphi$.*
- *$\mathcal{T}'$ respects implication, i.e. $\varphi \dot{\to} \psi \in \mathcal{T}'$ iff $\varphi \in \mathcal{T}' \to \psi \in \mathcal{T}'$.*
- *$\mathcal{T}'$ respects universal quantification, i.e. $\dot{\forall} \varphi \in \mathcal{T}'$ iff $\forall t. \varphi[t] \in \mathcal{T}'$.*

Extension in three subsequent steps:

0. $\mathcal{E} \supseteq \mathcal{T}$ which is exploding, i.e. $(\varphi_\bot \dot{\to} \varphi) \in \mathcal{E}$ for all closed φ:

$$\mathcal{E} := \mathcal{T} \cup \{\varphi_\bot \dot{\to} \varphi \mid \varphi \text{ closed}\}$$

1. $\mathcal{H} \supseteq \mathcal{E}$ which is Henkin, i.e. $(\varphi_n[n] \dot{\to} \dot{\forall} \varphi_n) \in \mathcal{H}$ for all n.
2. $\Omega \supseteq \mathcal{H}$ which is maximal, i.e. $\varphi \in \Omega$ if $\Omega, \varphi \vdash_c \varphi_\bot$ implies $\Omega \vdash_c \varphi_\bot$.

$\Rightarrow$ If $\mathcal{T} \nvdash_c \varphi_\bot$, then its extension $\mathcal{T}'$ resembles a syntactic model of $\mathcal{T}$.

Kripke Semantics

A Kripke model $\mathcal{K}$ over a domain D is a preorder $(\mathcal{W}, \preceq)$ with

$$f^{\mathcal{K}} : D^{|f|} \to D \qquad P^{\mathcal{K}} : \mathcal{W} \to D^{|P|} \to \mathbb{P} \qquad \bot^{\mathcal{K}} : \mathcal{W} \to \mathbb{P}$$

where we require $P_v^{\mathcal{K}}\,\vec{a} \to P_w^{\mathcal{K}}\,\vec{a}$ and $\bot_v^{\mathcal{K}} \to \bot_w^{\mathcal{K}}$ whenever $v \preceq w$.

Assignments ρ are extended to formulas $\mathbb{F}^*$:

$$w \Vdash_\rho \dot{\bot} := \bot_w^{\mathcal{K}} \qquad w \Vdash_\rho \varphi \dot{\to} \psi := \forall v \succeq w.\, v \Vdash_\rho \varphi \to v \Vdash_\rho \psi$$
$$w \Vdash_\rho P\,\vec{t} := P_w^{\mathcal{K}}\,(\hat{\rho}\,\vec{t}) \qquad w \Vdash_\rho \dot{\forall}\,\varphi := \forall a : D.\, w \Vdash_{a;\rho} \varphi$$

$\mathcal{K}$ is standard if $\bot_w^{\mathcal{K}} \to \bot$ for all w and exploding if $\mathcal{K} \Vdash \dot{\bot} \dot{\to} \varphi$ for all φ.
We write $\mathcal{T} \Vdash \varphi$ if $\mathcal{K} \Vdash_\rho \varphi$ for all standard $\mathcal{K}$ and ρ with $\mathcal{K} \Vdash_\rho \mathcal{T}$,
and $\mathcal{T} \Vdash_e \varphi$ when relaxing to exploding models.

Kripke Semantics: Semantic Cut-Elimination[10]

Introduce the cut-free sequent calculus LJT with focusing $\Gamma\,;\varphi \Rightarrow \psi$:

$$\frac{}{\Gamma\,;\varphi \Rightarrow \varphi}\ \mathsf{A} \qquad \frac{\Gamma\,;\varphi \Rightarrow \psi \quad \varphi \in \Gamma}{\Gamma \Rightarrow \psi}\ \mathsf{C} \qquad \frac{\Gamma \Rightarrow \varphi \quad \Gamma\,;\psi \Rightarrow \theta}{\Gamma\,;\varphi \dot{\to} \psi \Rightarrow \theta}\ \mathsf{IL}$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \dot{\to} \psi}\ \mathsf{IR} \qquad \frac{\Gamma\,;\varphi[t] \Rightarrow \psi}{\Gamma\,;\dot{\forall}\varphi \Rightarrow \psi}\ \mathsf{AL} \qquad \frac{\uparrow\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \dot{\forall}\varphi}\ \mathsf{AR} \qquad \frac{\Gamma \Rightarrow \dot{\bot}}{\Gamma \Rightarrow \varphi}\ \mathsf{E}$$

The following ingredients yield a cut-elimination procedure:

- $\Gamma \vdash \varphi$ implies $\Gamma \Vdash_e \varphi$ (straightforward by induction)
- $\Gamma \Vdash_e \varphi$ implies $\Gamma \Rightarrow \varphi$ (exploiting a universal model)
- Proofs $\Gamma \Rightarrow \varphi$ translate to normal proofs $\Gamma \vdash \varphi$ (by induction)

[10] Herbelin/Lee '09

Kripke Semantics: Constructive Completeness

The exploding model $\mathcal{U}$ over the domain $\mathbb{T}$ of terms is defined on the type of contexts Γ preordered by inclusion $\subseteq$. Further, we set:

$$f^{\mathcal{U}}\,\vec{d} := f\,\vec{d} \qquad P^{\mathcal{U}}_{\Gamma}\,\vec{d} := \Gamma \Rightarrow P\,\vec{d} \qquad \bot^{\mathcal{U}}_{\Gamma} := \Gamma \Rightarrow \dot{\bot}$$

We verify $\mathcal{U}$ following the normalisation-by-evaluation structure.[11]

Lemma

In the universal Kripke model $\mathcal{U}$ the following hold.

1. $\Gamma \Vdash_{\sigma} \varphi \to \Gamma \Rightarrow \varphi[\sigma]$
2. $(\forall\,\Gamma'\psi.\ \Gamma \subseteq \Gamma' \to \Gamma'\,;\varphi[\sigma] \Rightarrow \psi \to \Gamma' \Rightarrow \psi) \to \Gamma \Vdash_{\sigma} \varphi$

Corollary

$\Gamma \Vdash_{e} \varphi$ *implies* $\Gamma \Rightarrow \varphi$.

[11] Berger/Schwichtenberg '91

Kripke Semantics: Standard Completeness

The standard model $\mathcal{C}$ over the domain $\mathbb{T}$ of terms is defined on the type of consistent contexts $\Gamma \not\Rightarrow \dot{\bot}$ preordered by inclusion $\subseteq$. Further, we set:

$$f^{\mathcal{C}}\,\vec{d} := f\,\vec{d} \qquad P^{\mathcal{C}}_{\Gamma}\,\vec{d} := \neg\neg(\Gamma \Rightarrow P\,\vec{d}) \qquad \bot^{\mathcal{C}}_{\Gamma} := \bot$$

Lemma

In the universal Kripke model $\mathcal{C}$ the following hold.

1. $\Gamma \Vdash_\sigma \varphi \to \neg\neg(\Gamma \Rightarrow \varphi[\sigma])$
2. $(\forall\,\Gamma'\psi.\ \Gamma \subseteq \Gamma' \to \Gamma' ; \varphi[\sigma] \Rightarrow \psi \to \neg\neg(\Gamma' \Rightarrow \psi)) \to \Gamma \Vdash_\sigma \varphi$

Corollary

$\Gamma \Vdash \varphi$ *implies* $\neg\neg(\Gamma \Rightarrow \varphi)$.

As for $\Gamma \vdash_c \varphi$, completeness of $\Gamma \vdash \varphi$ is admissible and equivalent to $\mathsf{MP_L}$!