Computational Perspectives on Logical Foundations

Dominik Kirst

Application talk Inria Université Côte d'Azur, Stamp team

Past Activities

CV Overview

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Saarland University (2011-2015)

Bachelor's thesis with Gert Smolka

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Minerva Fellow with Liron Cohen

Inria/IRIF, Picube (since 2024)

Marie Curie Fellow with Hugo Herbelin

Three Overlapping Research Fields

Coq Proof Assistant

Dependent Type Theory

Constructive Mathematics

Field 1: Coq (Rocq) Proof Assistant

- Mechanised mathematics: four colour theorem, odd-order theorem, value of BB(5), ...
- Verification of realistic software: CompCert, CertiCoq, Iris, ...
- Applicable in teaching logic, arithmetics, calculus, ...

Contribution 1: Coq Library for Mechanised First-Order Logic

mark-koch Fix Proofmode MinZF demo ✓ 2722b67 4 days ago **History**

..		
Arithmetics	Rename Deduction -> ND to prepare for Sequent	2 months ago
Completeness2	Start using Asimpl in places	5 days ago
Deduction	Finish Kripke Completeness, work on atom substitution	10 days ago
Incompleteness	Tarski Constructions ported	2 months ago
Proofmode	Fix Proofmode MinZF demo	4 days ago
Reification	Tarski Constructions ported	2 months ago
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- Tool support for common implementation challenges and external usability

Field 2: Dependent Type Theory

- Richly typed functional programming language
- Interpretation as logical system base for proof assistants
- Natural connection of computation and logic

Contribution 2: Synthetic Approach to Computability Theory

Definition (Traditional)

A set $X \subseteq \mathbb{N}$ is decidable if there is a function $f : X \to \mathbb{B}$ such that

$$x \in X \;\leftrightarrow\; f(x) = \mathsf{tt}$$

and there is a total Turing machine M computing f: $\forall x.\, f(x) = \mathsf{tt} \leftrightarrow M[x] \downarrow 1$.

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- **Incompleteness**: PA/ZF (ITP'21), Tennenbaum's thm. (FSCD'22), abstraction (CSL'23)
- **Oracles**: definitions (APLAS'23), Post's thm. (CSL'24), Post's problem (TYPES'24)

Field 3: Constructive Mathematics

- Alternative foundation of mathematics based on more modest assumptions
  - Unveils hidden building blocks of mathematical constructions
    - Generalises connection of computation and logic

Contribution 3: Constructive Analysis of the Completeness Theorem

Theorem

If a first-order formula φ is valid in all models ($\vDash \varphi$) then it is provable ($\vdash$).

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- Generalisation (CPP'24) and semantic clarification (LICS'24)

Publications Overview

- Conference papers (APLAS, CPP, CSL, FSCD, IJCAR, ITP, LFCS, LICS, WoLLIC, 21 in total)
- Journal articles (JAR, LMCS, MSCS, Journal of Logic and Computation, 7 in total)
- Workshop abstracts (AAL, ALC, CCC, CoqPL, CoqWS, HoTT/UF, TYPES, 17 in total)
- International research network with 20 co-authors in total
- All projects are accompanied by Coq developments

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Visibility led to invited seminar talks, invited lectures at summer schools, service on PCs, features on a podcast

Software Development

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Coq Library for First-Order Logic (Leader):

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- 11 contributors and more than 25k loc

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Coq Library for Synthetic Computability (Contributor):

- Synthetic development of computability theory
- 7 contributors and more than 20k loc

Teaching and Supervision Experience

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At Saarland University:

- Student TA in several courses already during undergraduate studies
- Main organiser of basic and advanced lectures (up to 600 students)
- Conception of own seminars for specialised topics
- 11 supervised bachelor's students
- 2 supervised master's students
- 3 supervised engineering internships

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At Inria Paris:

- 1 supervised M2 internship
- 1 ongoing M2 internship
- Outreach via SIGPLAN-M mentoring program

Research Project:
Unified Reverse Mathematics

State of the Art

State of the Art

Classical reverse mathematics

Understand the logical strength of mathematical theorems

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Axioms $\Rightarrow$ Theorems

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$$\frac{\text{In principle shared motivation but}}{\text{incompatible base systems, different evaluation dimensions, sociological misunderstandings}}$$

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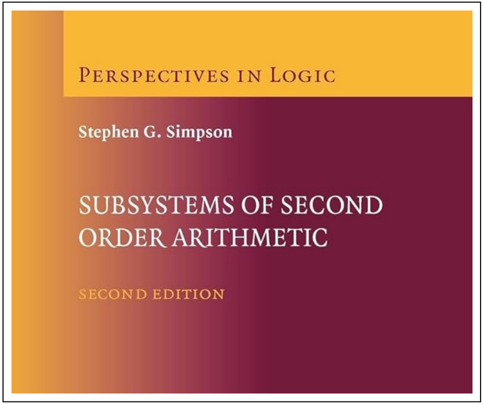
Understand the computational strength of mathematical theorems

State of the Art



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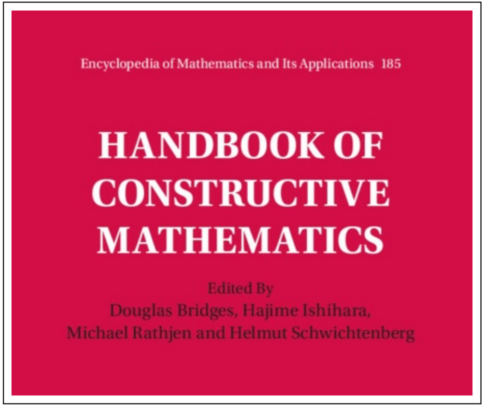
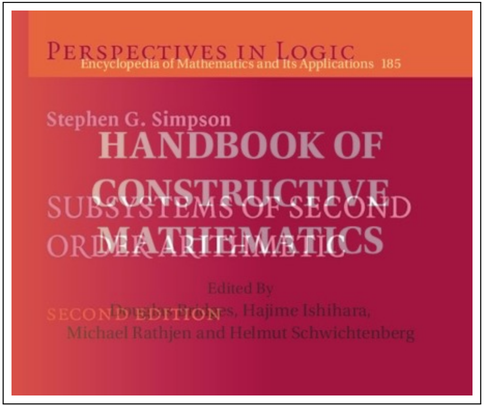


Constructive Reverse Mathematics

Vision



Unified Reverse Mathematics

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Axioms/Computations $\Leftrightarrow$ Theorems

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- **Constructive type theories like CIC**
  Embodying agnostic logical foundations suitable for both traditions,
  Integrating both logical and computational perspectives

Vision

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The previous challenges can be overcome by a combination of:

- **Constructive type theories like CIC**
  Embodying agnostic logical foundations suitable for both traditions,
  Integrating both logical and computational perspectives

- **Proof assistants like Coq implementing CIC**
  Providing support in the exploration and verification of new results,
  Implementing ideal tools for reverse mathematics

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3. Develop a proof assistant (variant) for the respective base systems
4. Implement a comprehensive formal library in it
5. Clarify the logical foundations of the respective base systems

Impact

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Theoretical impact:

- Long-sought reconciliation of classical and constructive reverse mathematics
- Deeper computational understanding of logical foundations
- Axiom minimisation generalises theorems to novel applications

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Technological impact:

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- Extensions of meta-programming tools like Elpi and MetaCoq
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Dissemination:

- Theory-oriented venues like CSL, FSCD, LICS, WoLLIC
- Mechanisation-focused venues like CPP, IJCAR, ITP, JAR

Integration at Inria

Integration with Stamp

Verification of algorithms and mathematical results:
Proof assistants, formal libraries, meta-programming, . . .

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- Yves Bertot: one of the main contributors to Coq, coordinator of LiberAbaci
- Enrico Tassi: formal libraries like MathComp and meta-programming in Elpi
- Laurent Théry: mechanised mathematics in MathComp Analysis

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- Laurent Théry: mechanised mathematics in MathComp Analysis
- Dominik Kirst: reverse mathematics, models of type theory, mechanised comp. physics

External Collaborations

- Aix-Marseille University *(France)*: Étienne Miquey
- Australian National University *(Australia)*: Ian Shillito
- Ben-Gurion University *(Israel)*: Liron Cohen, Ariel Grunfeld
- Birmingham University *(United Kingdom)*: Vincent Rahli, Bruno da Rocha Paiva
- ETH Zürich *(Switzerland)*: Johannes Hostert
- Gothenburg University *(Sweden)*: Dominik Wehr
- Inria Paris *(France)*: Hugo Herbelin, Yannick Forster
- Lorraine University *(France)*: Dominique Larchey-Wendling
- Max-Planck Institute SWS *(Germany)*: Niklas Mück, Benjamin Peters
- Oxford University *(United Kingdom)*: Janis Bailitis, Christian Hagemeier, Mark Koch
- Radboud University *(Netherlands)*: Marc Hermes
- Saarland University *(Germany)*: Gert Smolka, Haoyi Zeng
- TU Dortmund *(Germany)*: Andrej Dudenhefner

Summary

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Coq Proof Assistant

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$\Downarrow$

Research project: Unified Reverse Mathematics

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⇓

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Theoretical research (28 full papers, 20 international co-authors)

Software development (Coq library for first-order logic, tool support)

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FOL Library: Proof Mode

```
    frewrite (ax_add_zero y).
    fapply ax_refl.
  - fintros "x" "IH" "y".
    frewrite (ax_add_rec (σ y) x).
    frewrite ("IH" y).
    frewrite (ax_add_rec y x). fapply ax_refl.
Qed.

Lemma add_comm :
  FAI ⊢ << ∀' x y, x ⊕ y == y ⊕ x.
Proof.
  fstart. fapply ((ax_induction (<< Free x, ∀' y, x ⊕ y == y ⊕ x))).
  - fintros.
    frewrite (ax_add_zero x).
    frewrite (add_zero_r x).
    fapply ax_refl.
  - fintros "x" "IH" "y".
    frewrite (add_succ_r y x).
    frewrite <- ("IH" y).
    frewrite (ax_add_rec y x).
    fapply ax_refl.
Qed.

Lemma pa_eq_dec :
  FAI ⊢ << ∀' x y, (x == y) ∨ ¬ (x == y).
Proof.
  fstart.
  fapply ((ax_induction (<< Free x, ∀' y, (x == y) ∨ ¬ (x == y)))).
```

```
1 goal
p : peirce
x, y : term
______________________________________(1/1)
 FAI
"IH" : ∀  x0, x`[↑] ⊕  x0 ==  x0 ⊕ x`[↑]
━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━
(σ x ⊕ y == y ⊕ σ x)
```

Messages ↗ Errors ↗ Jobs ↗

https://github.com/dominik-kirst/coq-library-undecidability/blob/fol-library/theories/FOL/Proofmode/DemoPA.v

FOL Library: Reification Tactic

```
Proof.
elim a using PA_induction.
- represent.
- eapply ieq_trans. 1:apply (add_zero_l (iS b)).
  apply ieq_congr_succ, ieq_sym, add_zero_l.
- intros d IH.
  eapply ieq_trans. 1:apply (add_succ_l d (iS b)).
  apply ieq_congr_succ. eapply ieq_trans.
  + apply IH.
  + apply ieq_sym, add_succ_l.
Qed.

Lemma add_comm a b : a i⊕ b i= b i⊕ a.
Proof.
elim a using PA_induction.
- represent.
- eapply ieq_trans.
  + apply (add_zero_l b).
  + apply ieq_sym, (add_zero_r b).
- intros a' IH.
  eapply ieq_trans. 2:eapply ieq_trans.
  + apply (add_succ_l a' b).
  + apply ieq_congr_succ, IH.
  + apply ieq_sym, add_succ_r.
Qed.
```

```
1 goal
D' : Type
I : interp D'
D_fulfills : forall (f : form) (rho : env D'),
             PAeq f -> rho ⊨ f
a, b : D'
______________________________________(1/1)
representableP 1 (fun a0 : D => a0 i⊕ b i= b i⊕ a0)
```

Messages Errors Jobs

https://github.com/dominik-kirst/coq-library-undecidability/blob/fol-library/theories/FOL/Reification/DemoPA.v

Abstract Formal Systems

Definition

A formal system $\mathcal{S} = (\mathbb{S}, \vdash, \neg)$ consists of:

- $\mathbb{S}$ is a set we consider the collection of formal sentences
- $\vdash$ is a semi-decidable subset we consider the provable sentences
- $\neg : \mathbb{S} \to \mathbb{S}$ is a computable function we consider negation, satisfying consistency:

$$\forall \varphi \in \mathbb{S}.\ \nvdash \varphi \vee \ \nvdash \neg\varphi$$

$\mathcal{S}$ is complete if for every φ either $\vdash \varphi$ or $\vdash \neg\varphi$.

Lemma (Refutation)

In a complete formal system we have $\nvdash \varphi$ *iff* $\vdash \neg\varphi$.

Proof.

That $\nvdash \varphi$ implies $\vdash \neg\varphi$ is by completeness, the other direction by consistency. □

Gödel à la Turing

Theorem

Let X be some undecidable set, for instance the halting problem K_Θ. If X reduces to the provable sentences $\vdash$ of a formal system $\mathcal{S} = (\mathbb{S}, \vdash, \neg)$, then $\mathcal{S}$ cannot be complete.

Proof.

1. Assume that X is undecidable and that $\mathcal{S}$ were complete.
2. Obtain that that $\nvdash \varphi$ iff $\vdash \neg\varphi$ by (Refutation).
3. Observe that the complement of $\vdash$ is semi-decidable.
4. Derive that $\vdash$ is decidable by (Post).
5. Conclude that X must be decidable by (Reduction).
6. Contradiction. □

Constructive Analysis of the Completeness Theorem

There are multiple dimensions at play:

- Syntax fragment (e.g., propositional, minimal, negative, full)
- Complexity of the context (e.g., finite, decidable, enumerable, arbitrary)
- Cardinality of the signature (e.g., countable, uncountable)
- Representation of the semantics (e.g., Boolean, decidable, propositional)

All contribute (not even independently) to the constructive status of completeness!