Semantic Cut-elimination for Intuitionistic First-order Logic

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We present a normalization by evaluation style method of cut-elimination for a fragment of intuitionistic first-order logic using Kripke models based on prior work by Herbelin and Lee [1]. The results have been fully verified using the Coq proof assistant.

1 Preliminaries

Following [2, 3], we consider a simple predicate logic with falsity, implication, and universal quantification. The connectives of the object logic are marked with a dot, such as $\dot{\rightarrow}$, so they can be easily distinguished from their meta-logical counterparts. The terms consist of a unary function f as well as variables and constants ranging over $\mathbf{N}$. While the cut-elimination procedure generalizes easily to other term languages, this selection is exemplary for the features usually found in formulations of first-order logic. As we employ de Buijn style binders, the universal quantifier does not explicitly introduce a binding variable. As usual in intuitionistic systems, we will write $\dot{\neg}\varphi$ for $\varphi \dot{\rightarrow} \dot{\perp}$.

$$
\begin{array}{rcllr}
s, t : \mathsf{T} & ::= & c \mid f\ t \mid x & x : \mathbf{N}, c : \mathbf{N} & \textbf{terms} \\
\varphi, \psi : \mathsf{F} & ::= & \dot{\perp} \mid P\ s\ t \mid \varphi \dot{\rightarrow} \psi \mid \dot{\forall}\varphi & x : \mathbf{N} & \textbf{formulas}
\end{array}
$$

We follow Herbelin and Lee [1] by employing a Kripke semantics. However, we admit our models to have exploding nodes [4], which allows for a fully constructive completeness proof. An **interpretation I on domain D** is characterized by ($c^{\mathbf{I}} : \mathbf{N} \rightarrow \mathbf{D}$, $f^{\mathbf{I}} : \mathbf{D} \rightarrow \mathbf{D} \rightarrow \mathbf{D}$). Together with an **assignment** $\rho : \mathbf{N} \rightarrow \mathbf{D}$, it gives rise to a **term interpretation** $\cdot^{\mathbf{I},\rho} : \mathsf{T} \rightarrow \mathbf{D}$ as defined below. A **Kripke model on D** $\mathcal{K} = (\mathbf{I}, \mathbf{W}, \preccurlyeq, P_u, \perp_u)$ consists of an interpretation $\mathbf{I}$ on $\mathbf{D}$, a collection of worlds $\mathbf{W}$, a preorder $\preccurlyeq$ on $\mathbf{W}$ and interpretations $P_. : \mathbf{W} \rightarrow \mathbf{D} \rightarrow \mathbf{D} \rightarrow \mathbf{P}$, $\perp_. : \mathbf{W} \rightarrow \mathbf{P}$ which are monotonic with regards to $\preccurlyeq$. We define **formula satisfaction in a world** u via an embedding into our meta-logic $\vDash_u : (\mathbf{N} \rightarrow \mathbf{D}) \rightarrow \mathsf{F} \rightarrow \mathbf{P}$. We write $A \vDash \varphi$ if $(\forall \psi \in A.\ \rho \vDash_u \psi) \rightarrow \rho \vDash_u \varphi$ in all nodes u of all Kripke models $\mathcal{K}$ and under all assignments ρ.

$$
\begin{aligned}
\cdot^{\mathbf{I},\rho} &: \mathsf{T} \to \mathbf{D} \\
c^{\mathbf{I},\rho} &= c^{\mathbf{I}}\ c \\
(f\ t)^{\mathbf{I},\rho} &= f^{\mathbf{I}}\ t^{\mathbf{I},\rho} \\
x^{\mathbf{I},\rho} &= \rho\ x
\end{aligned}
\qquad\qquad
\begin{aligned}
\vDash_u &: (\mathbf{N} \to \mathbf{D}) \to \mathsf{F} \to \mathbf{P} \\
\rho \vDash_u \dot{\bot} &= \bot_u \\
\rho \vDash_u P\ s\ t &= P_u\ s^{\mathbf{I},\rho}\ t^{\mathbf{I},\rho} \\
\rho \vDash_{\mathbf{I}} \varphi \dot{\to} \psi &= \forall u \preccurlyeq v.\ \rho \vDash_v \varphi \to \rho \vDash_v \psi \\
\rho \vDash_{\mathbf{I}} \dot{\forall} x. \varphi &= \forall d : \mathbf{D}.\ \rho, d \vDash_{\mathbf{I}} \varphi
\end{aligned}
$$

As we are going to eliminate cuts, we first need a way of representing derivations that have cuts. For this purpose we use a standard **intuitionistic natural deduction system** which we denote with $\vdash_{ND} : \mathcal{L}(\mathsf{F}) \to \mathsf{F} \to \mathbf{P}$. Additionally, we use the cut-free **intuitionistic sequent calculus LJ** $: \mathcal{L}(\mathsf{F}) \to \mathcal{O}(\mathsf{F}) \to \mathsf{F} \to \mathbf{P}$ to represent the proofs after cut-elimination, which is given below. We write $A \vdash_{LJ} \varphi$ for $LJ\ A\ \emptyset\ \varphi$ and $A\,;\varphi \vdash_{LJ} \psi$ for $LJ\ A\ \varphi^o\ \psi$. Note that a proof in LJ can be turned back into a cut-free proof in ND in a straightforward manner. Here $\varphi[t]$ denotes the de Bruijn substitution of t into φ, $\uparrow\!A$ the shifting of all de Bruijn indices in A by 1 and A, φ the context A extended with φ.

$$
\textsc{Ctx}\,\frac{A\,;\varphi \vdash_{LJ} \psi \quad \varphi \in A}{A \vdash_{LJ} \psi}
\qquad
\textsc{Ax}\,\frac{}{A\,;\varphi \vdash_{LJ} \varphi}
\qquad
\text{IL}\,\frac{A \vdash_{LJ} \varphi \quad A\,;\psi \vdash_{LJ} \chi}{A\,;\varphi \dot{\to} \psi \vdash_{LJ} \chi}
$$

$$
\text{IR}\,\frac{A, \varphi \vdash \psi}{A \vdash \varphi \dot{\to} \psi}
\qquad
\textsc{Exp}\,\frac{A \vdash \dot{\bot}}{A \vdash \varphi}
\qquad
\text{AL}\,\frac{A\,;\varphi[t] \vdash_{LJ} \psi}{A\,;\dot{\forall}\varphi \vdash_{LJ} \psi}
\qquad
\text{AR}\,\frac{\uparrow\!A \vdash_{LJ} \varphi}{A \vdash_{LJ} \dot{\forall}\varphi}
$$

There are multiple ways of formalizing the AR rule: the approach taken here is called "de Bruijn", a popular alternative is named "locally nameless". De Bruijn allows for pleasant weakening proofs while locally nameless will prove very useful during the completeness proof. The following lemma allows us to freely switch between the two representations.

Theorem 1 (De Bruijn and locally nameless equivalence) Let A be a context and φ be a formula. Further, let x be a variable that is not bound in A or φ. Then

$$
\uparrow\!A \vdash_{LJ} \varphi \quad \text{iff} \quad A \vdash_{LJ} \varphi[x] \qquad\qquad \uparrow\!A \vdash_{ND} \varphi \quad \text{iff} \quad A \vdash_{ND} \varphi[x]
$$

Proof By applying the correct substitutions to the proofs. ∎

2 Semantic cut-elimination

We mostly follow the approach of Herbelin and Lee [1]. It employs the well known **normalization by evaluation** method of cut-elimination [5, 6]. Given a language L, one defines three functions: the **interpretation** $[\![\cdot]\!] : \mathsf{L} \to \mathcal{D}$ which translates terms into a computational denotational semantics and the mutually recursive **reflect** $\uparrow : \mathsf{L}^n \to \mathcal{D}$ and **reify** $\downarrow : \mathcal{D} \to \mathsf{L}^n$ which translate the denotational semantics back into normal terms. A term t can then be normalized by reifying its interpretation $\downarrow [\![t]\!]$.

In the case of this proof, we will use Kripke models for the semantics, a soundness proof as the interpretation function and the reification will be performed by a completeness proof. We will make use of a special Kripke model, known as the universal model, during the completeness proof.

Theorem 2 (Soundness) Let A be a context and φ a formula. Then $A \vdash_{ND} \varphi \to A \vDash \varphi$.

Proof By induction on $A \vdash_{ND} \varphi$. ■

Definition 3 (The universal model) The universal Kripke model $\mathcal{U}$ is given by the quintuple $(\mathbf{I}, \mathcal{L}(\mathsf{F}), \subseteq, P_A, \bot_A)$, where

- $\mathbf{I}$ denotes the identity interpretation with $t^{\mathbf{I},\rho} = t[\rho]$ for all terms t and substitutions ρ
- $\mathcal{L}(\mathsf{F})$ describes the collection of all contexts
- $\subseteq$ is the subset relation on contexts
- we choose $P_A\ s\ t := A \vdash_{LJ} P\ s\ t$ and $\bot_A := A \vdash_{LJ} \bot$

Theorem 4 (Reify and Reflect) Let A be a context, φ a formula and ρ a substitution. Within the universal model $\mathcal{U}$

(i) $\rho \vDash_A \varphi \to A \vdash_{LJ} \varphi[\rho]$

(ii) $(\forall B\, \psi.\ A \subseteq B \to\ \ B\,; \varphi[\rho] \vdash_{LJ} \psi\ \ \to B \vdash_{LJ} \psi) \to \rho \vDash_A \varphi$

Proof Simultaneous proof by induction on the formula φ of the generalized statements

(i) $\forall A\, \rho.\ \rho \vDash_A \varphi \to A \vdash_{LJ} \varphi[\rho]$

(ii) $\forall A\, \rho.\ (\forall B\, \psi.\ A \subseteq B \to\ \ B\,; \varphi[\rho] \vdash_{LJ} \psi\ \ \to B \vdash_{LJ} \psi) \to \rho \vDash_A \varphi$

We only cover the cases of $\dot\forall$ and $\dot\to$, the rest is straightforward.

Case $\varphi = \dot\forall\varphi$: (i) Assuming $\forall t.\ \rho, t \vDash_A \varphi$, we have to show $A \vdash_{LJ} \dot\forall\varphi$. By AR, it suffices to show $\uparrow\!A \vdash_{LJ} \varphi[\Uparrow\!\rho]$ which by Theorem 1 is equivalent to $A \vdash_{LJ} \varphi[\rho, x]$ for a free variable x. This holds per our initial assumption and inductive hypothesis (i).

(ii) Assuming $\forall B\, \chi.\ A \subseteq B \to B\,; (\dot\forall\varphi)[\rho] \vdash_{LJ} \chi \to B \vdash_{LJ} \chi$, we have to show $\rho, t \vDash_A \varphi$ for every term t. Per inductive hypothesis (ii) it suffices to show $\forall B\, \chi.\ A \subseteq B \to B\,; \varphi[\rho, t] \vdash_{LJ} \chi \to B \vdash_{LJ} \chi$. Using our assumption, we still have to deduce $B\,; (\dot\forall\varphi)[\rho] \vdash_{LJ} \chi$ from these premises, which can be achieved via AL.

Case $\varphi = \varphi \dot{\rightarrow} \psi$: (i) Assuming $\forall B.\ A \subseteq B \rightarrow \rho \vDash_B \varphi \rightarrow \rho \vDash_B \psi$, we have to show $A \vdash_{LJ} (\varphi \dot{\rightarrow} \psi)[\rho]$. Per IR and inductive hypothesis (i) for ψ it suffices to show $\rho \vDash_{A,\varphi[\rho]} \psi$. Using the inductive hypothesis (ii) for φ and our assumption, it suffices to show $\forall B\ \chi.\ A, \varphi[\rho] \subseteq B \rightarrow B\,;\varphi[\rho] \vdash_{LJ} \chi \rightarrow B \vdash_{LJ} \chi$, which holds per Ax.

(ii) Assuming $\forall B\ \chi.\ A \subseteq B \rightarrow B\,;(\varphi \dot{\rightarrow} \psi)[\rho] \vdash_{LJ} \chi \rightarrow B \vdash_{LJ} \chi$ we have to show $\forall B.\ A \subseteq B \rightarrow \rho \vDash_B \varphi \rightarrow \rho \vDash_B \psi$. Because of the inductive hypothesis (ii) for ψ it suffices to show $\forall C\ \chi.\ B \subseteq C \rightarrow C\,;\psi[\rho] \vdash_{LJ} \chi \rightarrow C \vdash_{LJ} \chi$, which, using our assumption, turns into $C\,;(\varphi \dot{\rightarrow} \psi)[\rho] \vdash_{LJ} \chi$. This follows using IL and inductive hypothesis (i) for φ. ■

Corollary 5 (Completeness) Let A be a context and φ a formula. Then $A \vDash \varphi \rightarrow A \vdash_{LJ} \varphi$.

Corollary 6 (Cut-elimination) For any context A and formula φ, $A \vdash_{ND} \varphi \rightarrow A \vdash_{LJ} \varphi$.

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